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Eulerian Walks

Flow Decomposition and Transformations

Eulerian Walks in Directed Graphs in O(m) time.

Step 1. Create a breadth first search tree into node 1. For j not equal to 1, put the arc out of j in T last on the arc list A(j).

Step 2. Create an Eulerian cycle by starting a walk at node 1 and selecting arcs in the order they appear on the arc lists.

Proof of Correctness

Relies on the following observation and invariant:

Observation: The walk will terminate at node 1. Whenever the walk visits node j for j ≠ 1, the walk has traversed one more arc entering node j than leaving node j.

Invariant: If the walk has not traversed the tree arc for node j, then there is a path from node j to node 1 consisting of nontraversed tree arcs.

Eulerian Cycle Animation

Eulerian Cycles in undirected graphs

Strategy: reduce to the directed graph problem as follows:

Step 1. Use dfs to partition the arcs into disjoint cycles

Step 2. Orient each arc along its directed cycle. Afterwards, for all i, the number of arcs entering node i is the same as the number of arcs leaving node i.

Step 3. Run the algorithm for finding Eulerian Cycles in directed graphs

Flow Decomposition and Transformations

- Flow Decomposition
- Removing Lower Bounds
- Removing Upper Bounds
- Node splitting

- **Arc flows**: an arc flow x is a vector x satisfying:

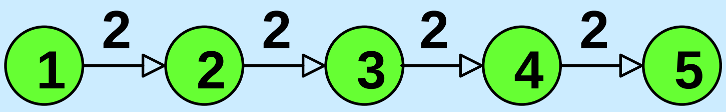
Let $b(i) = \sum_j x_{ij} - \sum_i x_{ji}$

We are not focused on upper and lower bounds on x for now.

Flows along Paths

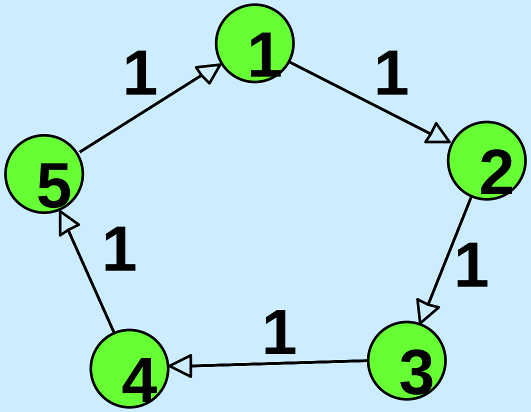
Usual: represent flows in terms of flows in arcs.

Alternative: represent a flow as the sum of flows in paths and cycles.



P

Two units of flow in the path P



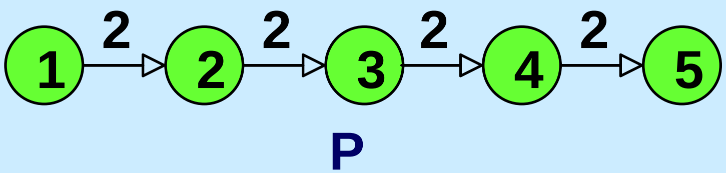
C

One unit of flow around the cycle C

Properties of Path Flows

Let P be a directed path.

Let ***Flow(™,P)*** be a flow of ™ units in each arc of the path P.

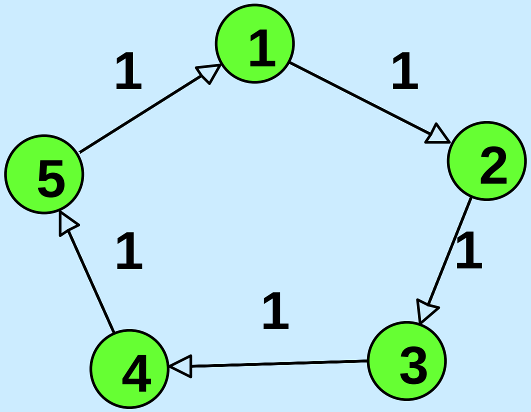


Flow(2, P)

Observation. If P is a path from s to t, then Flow(™,P) sends units of δ flow from s to t, and has conservation of flow at other nodes.

Property of Cycle Flows

- If p is a cycle, then sending one unit of flow along p satisfies conservation of flow everywhere.



Representations as Flows along Paths and Cycles

Let $\mathcal{P}$ be a collection of Paths; let f(P) denote the flow in path P

Let $\mathcal{C}$ be a collection of cycles; let f(C) denote the flow in cycle C.

One can convert the path and cycle flows into an arc flow x as follows: for each arc (i,j) $\in$ A

$$x_{ij} = \sum_{P \ni (i,j)} f(P) \quad + \sum_{C \ni (i,j)} f(C)$$

Flow Decomposition

x: Initial flow

y: updated flow

G(y): subgraph with arcs (i, j) with $y_{ij} > 0$ and incident nodes

f(P) Flow around path P (during the algorithm)

$\mathcal{P}$: paths with flow in the decomposition

$\mathcal{C}$: cycles with flow in the decomposition

INVARIANT

$$x_{ij} = y_{ij} + \sum_{P \ni (i,j)} f(P) + \sum_{C \ni (i,j)} f(C)$$

Initially, x = y and f = 0.

At end, y = 0, and f gives the flow decomposition.

Deficit and Excess Nodes

Let x be a flow (not necessarily feasible)

If the flow out of node i exceeds the flow into node i, then node i is a deficit node.

Its deficit is $\sum_j x_{ij} - \sum_k x_{ki}$.

If the flow out of node i is less than the flow into node i, then node i is an excess node.

Its excess is $-\sum_j x_{ij} + \sum_k x_{ki}$.

If the flow out of node i equals the flow into node i, then node i is a balanced node.

Flow Decomposition Algorithm

Step 0. Initialize: y := x; f := 0; $\mathcal{P}$:= ∅ ; $\mathcal{C}$:= ∅;

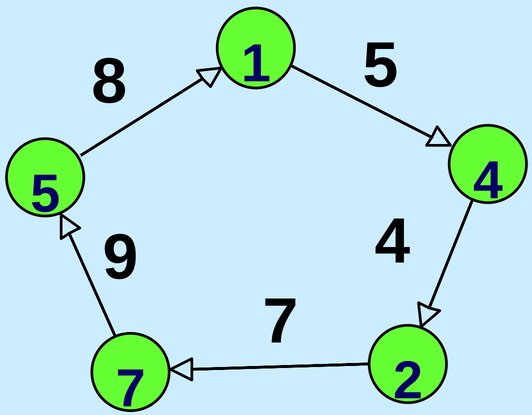
Step 1. Select a deficit node j in G(y). If no deficit node exists, select a node j with an incident arc in G(y);

Step 2. Carry out depth first search from j in G(y) until finding a directed cycle W in G(y) or a path W in G(y) from s to a node t with excess in G(y).

Step 3.

1. **Let Δ = capacity of W in G(y). (See next slide)**
2. **Add W to the decomposition with f(W) = Δ.**
3. **Update y (subtract flow in W) and excesses and deficits**
4. **If y ≠ 0, then go to Step 1**

Capacities of Paths and Cycles



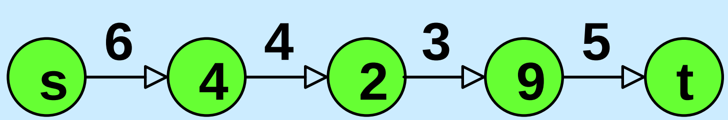
C

The capacity of C is = min arc flow on C wrt flow y.

capacity = 4

deficit = 3

excess = 2



P

The capacity of P is denoted as Δ(P, y) = min[def(s), excess(t), min (x_{ij} : (i,j) ∈ P)]

χαπαχιτψ = 2

Flow Decomposition Animation

Complexity Analysis

- ***Select initial node:***
  - **O(1) per path or cycle, assuming that we maintain a set of supply nodes and a set of balanced nodes incident to a positive flow arc**

- ***Find cycle or path***
  - **O(n) per path or cycle since finding the next arc in depth first search takes O(1) steps.**

- ***Update step***
  - **O(n) per path or cycle**

Complexity Analysis (continued)

Lemma. The number of paths and cycles found in the flow decomposition is at most m + n – 1.

Proof. In the update step for a cycle, at least one of the arcs has its capacity reduced to 0, and the arc is eliminated.

In an update step for a path, either an arc is eliminated, or a deficit node has its deficit reduced to 0, or an excess node has its excess reduced to 0.

(Also, there is never a situation with exactly one node whose excess or deficit is non-zero).

Conclusion

Flow Decomposition Theorem. **Any non-negative feasible flow x can be decomposed into the following:**

i. **the sum of flows in paths directed from deficit nodes to excess nodes, plus**

ii. **the sum of flows around directed cycles.**

It will always have at most n + m paths and cycles.

Remark. The decomposition usually is not unique.

Corollary

A *circulation* is a flow with the property that the flow in is the flow out for each node.

Flow Decomposition Theorem for circulations. **Any non-negative feasible flow x can be decomposed into the sum of flows around directed cycles.**

It will always have at most m cycles.

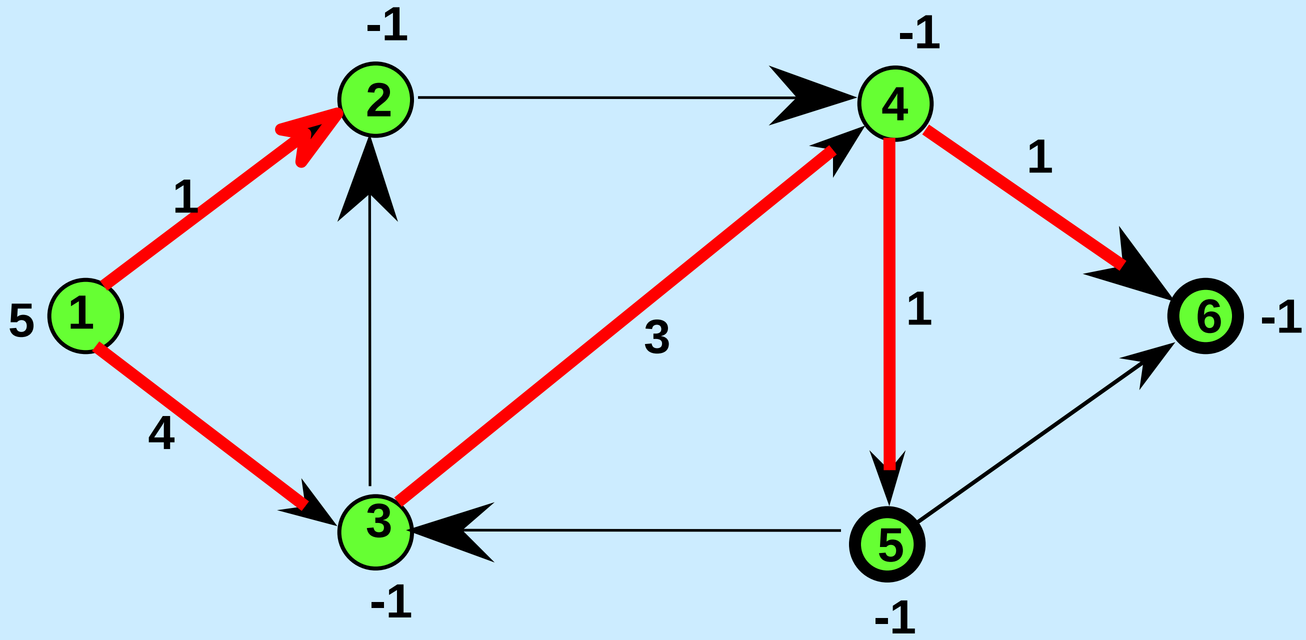
An application of Flow Decomposition

Consider a feasible flow where the supply of node 1 is n-1, and the supply of every other node is -1.

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} n-1 & if\ i=1 \\ -1 & if\ i \neq 1 \end{cases}$$

Suppose the arcs with positive flow have no cycle. Then the flow can be decomposed into unit flows along paths from node 1 to node j for each j ≠ 1.

A flow and its decomposition



The decomposition of flows yields the paths:

1-2,   1-3,   1-3-4

1-3-4-5   and 1-3-4-6.

There are no cycles in the decomposition.

Application to shortest paths

To find a shortest path from node 1 to each other node in a network, find a minimum cost flow in which b(1) = n-1 and b(j) = -1 for j ≠ 1.

The flow decomposition gives the shortest paths.

Other Applications of Flow Decomposition

- Reformulations of Problems.
  - There are network flow models that use path and cycle based formulations.
  - Multicommodity Flows
- Used in proving theorems
- Can be used in developing algorithms

The min cost flow problem (again)

The minimum cost flow problem

u_{ij} = capacity of arc (i,j).

c_{ij} = unit cost of flow sent on (i,j).

x_{ij} = amount shipped on arc (i,j)

Minimize $\sum c_{ij}x_{ij}$

$\sum_j x_{ij} - \sum_k x_{ki} = b_i$ for all $i \in N$.

and $0 \leq x_{ij} \leq u_{ij}$ for all $(i,j) \in A$.

The model seems very limiting

- The lower bounds are 0.
- The supply/demand constraints must be satisfied exactly
- There are no constraints on the flow entering or leaving a node.

We can model each of these constraints using transformations.

- In addition, we can transform a min cost flow problem into an equivalent problem with no upper bounds.

Eliminating Lower Bound on Arc Flows

Suppose that there is a lower bound l_{ij} on the arc flow in (i,j)

Minimize $\sum c_{ij}x_{ij}$

$$\sum_j x_{ij} - \sum_k x_{ki} = b_i \quad \text{for all } i \in N.$$

$$\text{and } l_{ij} \leq x_{ij} \leq u_{ij} \quad \text{for all } (i,j) \in A.$$

Then let $y_{ij} = x_{ij} - l_{ij}$. Then $x_{ij} = y_{ij} + l_{ij}$

Minimize $\sum c_{ij}(y_{ij} + l_{ij})$

$$\sum_j (y_{ij} + l_{ij}) - \sum_k (y_{ij} + l_{ij}) = b_i \quad \text{for all } i \in N.$$

$$\text{and } l_{ij} \leq (y_{ij} + l_{ij}) \leq u_{ij} \quad \text{for all } (i,j) \in A.$$

Then simplify the expressions.

Allowing inequality constraints

Minimize $\sum c_{ij}x_{ij}$

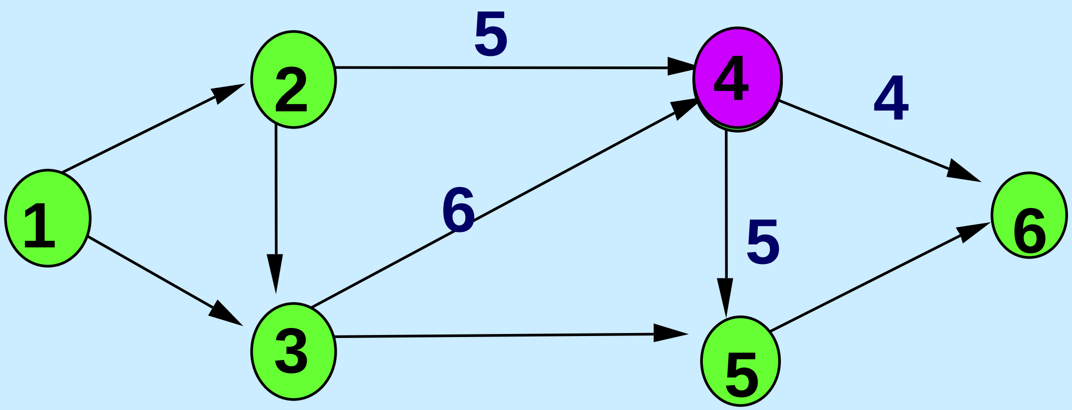
$\sum_j \; x_{ij} - \sum_k x_{ki} \le b_i$ for all $i \in N$.

and $l_{ij} \le x_{ij} \le u_{ij}$ for all $(i,j) \in A$.

Let $B = \sum_i b_i$. For feasibility, we need $B \ge 0$

Create a "dummy node" n+1, with $b_{n+1} = -B$. Add arcs (i, n+1) for i = 1 to n, with $c_{i,n+1} = 0$. Any feasible solution for the original problem can be transformed into a feasible solution for the new problem by sending excess flow to node n+1.

Node Splitting

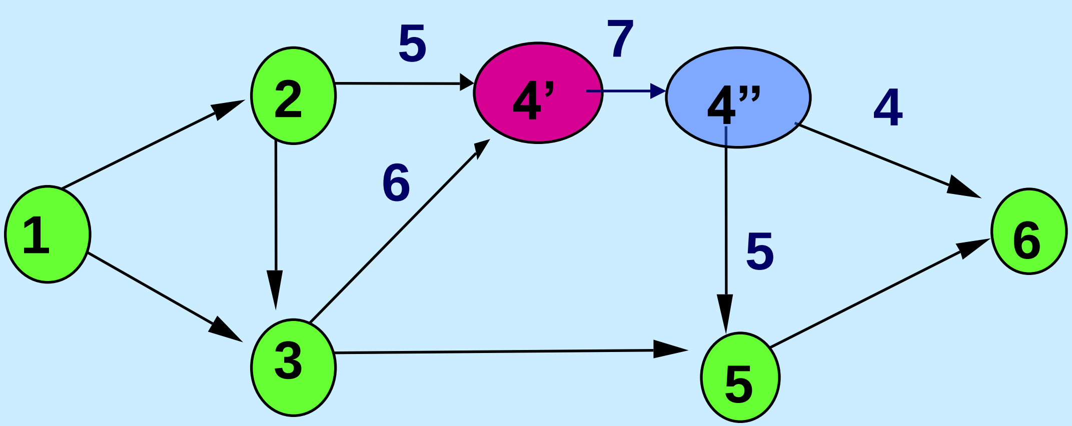


Flow x

Arc numbers are capacities

Suppose that we want to add the constraint that the flow into node 4 is at most 7.

Method: split node 4 into two nodes, say 4' and 4"



Flow x' can be obtained from flow x, and vice versa.

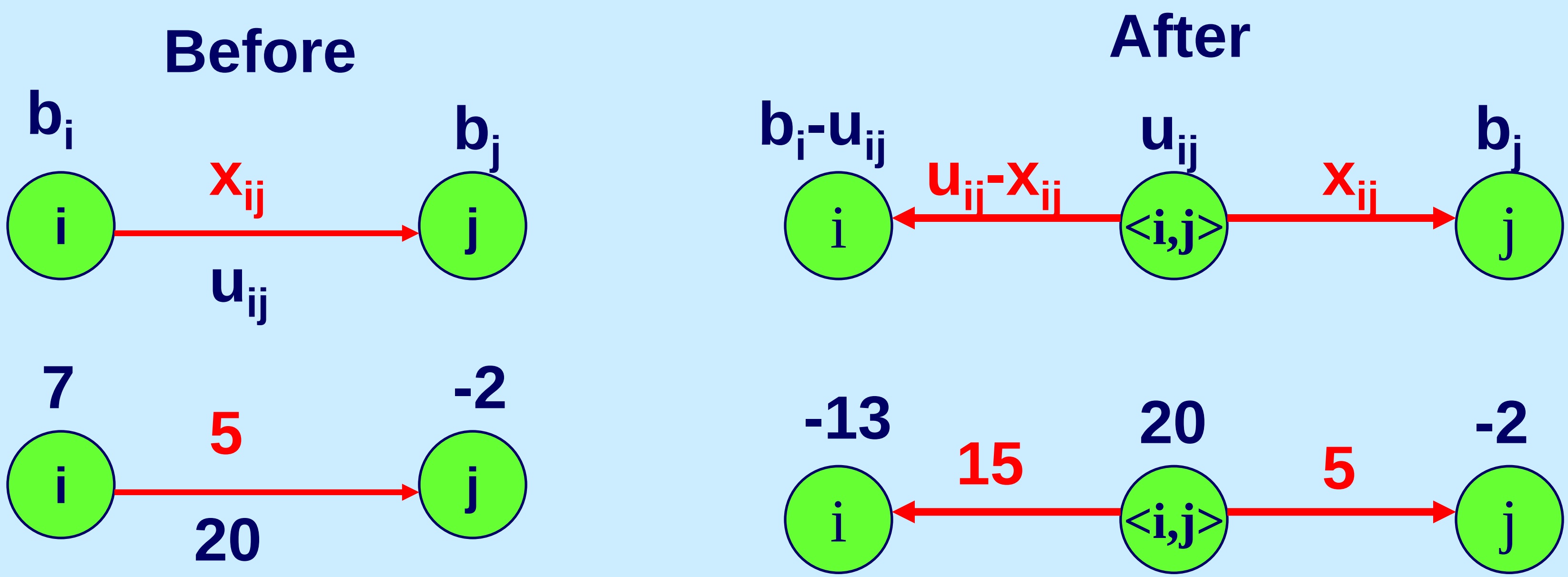
Eliminating Upper Bounds on Arc Flows

The minimum cost flow problem

Min $\sum c_{ij}x_{ij}$

s.t. $\sum_j x_i - \sum_k x_{ki} = b_i$ for all $i \in N$.

and $0 \le x_{ij} \le u_{ij}$ for all $(i,j) \in A$.



Summary

1. Efficient implementation of finding an eulerian cycle.

2. Flow decomposition theorem

3. Transformations that can be used to incorporate constraints into minimum cost flow problems.