**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**

6.265/15.070J — Fall 2013
Lecture 20 — 11/25/2013

## Introduction to the theory of weak convergence

**Content.**

1. $\sigma$-fields on metric spaces.
2. Kolmogorov $\sigma$-field on $C[0, T]$.
3. Weak convergence.

In the first two sections we review some concepts from measure theory on metric spaces. Then in the last section we begin the discussion of the theory of weak convergence, by stating and proving important Portmentau Theorem, which gives four equivalent definitions of weak convergence.

## 1 Borel $\sigma$-fields on metric space

We consider a metric space $(S, \rho)$. To discuss probability measures on metric spaces we first need to introduce $\sigma$-fields.

**Definition 1.** *Borel $\sigma$-field $\mathcal{B}$ on $S$ is the field generated by the set of all open sets $U \subset S$.*

**Lemma 1.** *Suppose $S$ is Polish. Then every open set $U \subset S$ can be represented as a countable union of balls $B(x, r)$.*

*Proof.* Since $S$ is Polish, we can identify a countable set $x_1, \ldots, x_n, \ldots$ which is dense. For each $x_i \in U$, since $U$ is open we can find a radius $r_i$ such that $B(x_i, r_i) \subset U$. Fix a constant $M > 0$. Consider $r_i^* = \min(\arg\sup\{r : B(x, r) \subset U\}, M) > 0$ and set $r_i = r_i^*/2$. Then $\cup_{i:x_i \in U} B(x_i, r_i) \subset U$. In order to finish the proof we need to show that equality $\cup_{i:x_i \in U} B(x_i, r_i) = U$ holds. Consider any $x \in U$. There exists $0 < r \leq M$ such that $B(x, r) \subset U$. Since set $(x_i)$ is dense there exists $x_k$ such that $\rho(x_k, x) < r/4$. Then, by triangle inequality, $B(x_k, 3r/4) \subset B(x, r) \subset U$. This means, by definition of $r_k^*$ and $r_k$, that $r_k \geq 3r/8$. But $x \in B(x_k, 3r/8)$ since the distance

$\rho(x_k, x) \leq r/4$. This shows $x \in B(x_k, r_k)$. Since $x$ was arbitrary, then $U \subset \cup_{i:x_i \in U} B(x_i, r_i)$. $\square$

From this Lemma we obtain immediately that

**Corollary 1.** *Suppose $S$ is Polish. Then $\mathcal{B}$ is generated by a countable collection of balls $B(x, r), x \in S, r \geq 0$.*

## 2 Kolmogorov $\sigma$-field on $C[0, T]$

Now let us focus on $C[0, T]$ and the Borel $\sigma$ field $\mathcal{B}$ on it. For each $t \geq 0$ define a projection $\pi_t : C[0, T] \to \mathbb{R}$ as $\pi_t(x) = x(t)$. Observe that $\pi_t$ is a uniformly continuous mapping. Indeed

$$|\pi_t(x) - \pi_t(y)| = |x(t) - y(t)| \leq \|x - y\|.$$

This immediately implies uniform continuity.

The family of projection mappings $\pi_t$ give rise to an alternative field.

**Definition 2.** *The Kolmogorov $\sigma$-field $\mathcal{K}$ on $C[0, T]$ is the $\sigma$-field generated by $\pi_t^{-1}(B), t \in [0, T], B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel field of $\mathbb{R}$.*

It turns out (and this will be useful) that the two fields are identical:

**Theorem 1.** *The Kolmogorov $\sigma$ field $\mathcal{K}$ is identical to the Borel field $\mathcal{B}$ of $C[0, T]$.*

*Proof.* First we show that $\mathcal{K} \subset \mathcal{B}$. Since $\pi_t$ is continuous then, for every open set $U \subset \mathbb{R}$ $\pi_t^{-1}(U)$ is open in $C[0, T]$. This applies to all open intervals $U$. Thus each $\pi_t^{-1}(U) \in \mathcal{B}$. This shows $\mathcal{K} \subset \mathcal{B}$.

Now we show the other direction. Since $C[0, T]$ is Polish, then by Corollary 1, it suffices to check that every ball $B(x, r) \in \mathcal{K}$. Fix $x \in C[0, T], r \geq 0$. For each rational $q \in [0, T]$, consider $B_q \triangleq \pi_q^{-1}([x(q) - r, x(q) + r])$. This is the set of all functions $y$ such that $|y(q) - x(q)| \leq r$. Consider $\cap_q B_q$. As a countable intersection, this is a set in $\mathcal{K}$. We claim that $\cap_q B_q = B$. This implies the result.

To establish the claim, note that $B \subset B_q$ for each $q$. Now suppose $y \notin B$. Namely, for some $t \in [0, T]$ we have $|y(t) - x(t)| \geq r + \delta > r$. Find a sequence of rational values $q_n$ converging to $t$. By continuity of $x, y$ we have $x(q_n) \to x(t), y(q_n) \to y(t)$. Therefore for all sufficiently large $n$ we have $|y(q_n) - x(q_n)| > r$. This means $y \notin \cap B_q$. $\square$

## 3 Weak convergence

We now turn to a very important concept of weak convergence or convergence of probability measures. Recall the convergence in distribution of r.v. $X_n \Rightarrow X$. Now consider random variables $X : \Omega \to S$ which take values in some metric space $(S, \rho)$. Again we define $X$ to be a random variable if $X$ is a measurable transformation. We would like to give a meaning to $X_n \Rightarrow X$. In order to do this we first define convergence of probability measures.

Given a metric space $(S, \rho)$ and the corresponding Borel $\sigma$-field $\mathcal{B}$, suppose we have a sequence $\mathbb{P}, \mathbb{P}_n, n = 1, 2, \ldots$ of probability measures on $(S, \mathcal{B})$. Let $C_b(S)$ denote the set of all continuous bounded real valued functions on $S$. In particular every function $X \in C_b(S)$ is measurable.

**Theorem 2** (**Portmentau Theorem**). *The following conditions are equivalent.*

1. $\lim_n \mathbb{E}_{\mathbb{P}_n}[X] = \mathbb{E}_{\mathbb{P}}[X], \forall X \in C_b(S)$.
2. *For every closed set* $F \subset S$, $\limsup_n \mathbb{P}_n(F) \le \mathbb{P}(F)$.
3. *For every open set* $U \subset S$, $\liminf_n \mathbb{P}_n(U) \ge \mathbb{P}(U)$.
4. *For every set* $A \in \mathcal{B}$ *such that* $\mathbb{P}(\partial A) = 0$, *the convergence* $\lim_n \mathbb{P}_n(A) = \mathbb{P}(A)$ *holds.*

What is the meaning of this theorem? Ideally we would like to say that a sequence of measures $\mathbb{P}_n$ converges to measure $\mathbb{P}$ if for every set $A$, $\mathbb{P}_n(A) \to \mathbb{P}(A)$. However, this turns out to be too restrictive. In some sense the fourth part of the theorem is the meaningful part. Second and third are technical. The first is a very useful implication.

**Definition 3.** *A sequence of measures* $\mathbb{P}_n$ *is said to converge weakly to* $\mathbb{P}$ *if one of the four equivalent conditions in Theorem 2 holds.*

*Proof.* (a)$\Rightarrow$(b) Consider a closed set $F$. For any $\epsilon > 0$, define $F^\epsilon = \{x \in S : \rho(x, S) \le \epsilon\}$, where the distance between the sets is defined as the smallest distance between any two points in them. Let us first show that $\cap_\epsilon F^\epsilon = F$. The inclusion $\cap_\epsilon F^\epsilon \supset F$ is immediate. For the other side, consider any $x \notin F$. Since $S \setminus F$ is open, then $B(x, r) \subset S \setminus F$ for some $r$. This means $x \notin F^r$ and the assertion is established.

Invoking the continuity theorem, the equality $\cap_\epsilon F^\epsilon = F$ implies

$$\lim_{\epsilon \to 0} \mathbb{P}(F^\epsilon) = \mathbb{P}(F). \tag{1}$$

Define $X_\epsilon : S \to [0,1]$ by $X_\epsilon(x) = (1-\rho(x,F)\epsilon^{-1})^+$. Then $X_\epsilon = 1$ on $F$ and $X_\epsilon = 0$ outside of $F^\epsilon$. Specifically,

$$1\{F\} \le X_\epsilon \le 1\{F^\epsilon\},$$

which implies

$$\mathbb{P}_n(F) \le \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] \le \mathbb{P}_n(F^\epsilon), \qquad \mathbb{P}(F) \le \mathbb{E}_{\mathbb{P}}[X_\epsilon] \le \mathbb{P}(F^\epsilon),$$

Note that $X_\epsilon$ is a continuous bounded function. Therefore, by assumption $\lim_n \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] = \mathbb{E}[X_\epsilon]$. Combining, we obtain that

$$\limsup_n \mathbb{P}_n(F) \le \limsup_n \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] = \lim_n \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] = \mathbb{E}_{\mathbb{P}}[X_\epsilon] \le \mathbb{P}(F^\epsilon)$$

Finally, using (1) we conclude that $\limsup_n \mathbb{P}_n(F) \le \mathbb{P}(F)$.

(b)$\Rightarrow$ (c) This part follows immediately by observing that $F = U^c$ is a closed set and $\mathbb{P}(F) = 1 - \mathbb{P}(U), \mathbb{P}_n(F) = 1 - \mathbb{P}_n(U)$. Moreover (c)$\Rightarrow$(b).

(c)$\Rightarrow$ (d) Given any set $A$, consider its closure $\bar{A} \supset A$ and interior $A^o \subset A$. Applying (b) and (c), we have

$$\mathbb{P}(\bar{A}) \ge \limsup_n \mathbb{P}_n(\bar{A}) \ge \limsup_n \mathbb{P}_n(A) \ge \liminf_n \mathbb{P}_n(A) \ge \liminf_n \mathbb{P}_n(A^o) \ge \mathbb{P}(A^o).$$

Now if $\mathbb{P}(\partial A) = \mathbb{P}(\bar{A} \setminus A^o) = 0$, then we obtain an equality across and, in particular, $\mathbb{P}_n(A) \to \mathbb{P}(A)$.

(d)$\Rightarrow$(a) Let $X \in C_b(S)$. We may assume w.l.g that $0 \le X \le 1$. Then $\mathbb{E}_{\mathbb{P}_n}[X] = \int_0^1 \mathbb{P}_n(X > t)dt, \mathbb{E}_{\mathbb{P}}[X] = \int_0^1 \mathbb{P}(X > t)dt$. Since $X$ is continuous, for any set $A(t) = \{x \in S : X(x) > t\}$, its boundary satisfies $\partial A \subset 1\{X = t\}$ (exercise). Observe that probability associated with set $1\{X = t\}$ is zero, except for countably many points $t$. This can be obtained by looking at sets of the form $1\{X = t\}$ with probability weights at least $1/m$ and taking a union over $m$. Thus for almost all $t$, the set $A(t)$ is a continuity set, i.e., $\mathbb{P}(\partial A) = 0$ and hence $\mathbb{P}_n(A(t)) \to \mathbb{P}(A(t))$. Since also $0 \le \mathbb{P}, \mathbb{P}_n \le 1$, then using the Bounded Convergence Theorem applied to "random" $t$, we obtain that

$$\int_0^1 \mathbb{P}_n(A(t))dt \to \int_0^1 \mathbb{P}(A(t))dt.$$

This implies the result. $\square$

Using the definition of weak convergence of measures we can define weak convergence of metric space valued random variables.

**Definition 4.** *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a metric space* $(S, \rho)$*, a sequence of measurable transformations* $X_n : \Omega \to S$ *is said to converge weakly or in distribution to a transformation* $X : \Omega \to S$*, if the probability measures on* $(S, \rho)$ *generated by* $X_n$ *converge weakly to the probability measure generated by* $X$ *on* $(S, \rho)$.

## 4 Additional reading materials

- Notes distributed in the class, Chapter 5.
- Billigsley [1] Chapter 1, Section 2.

## References

[1] P. Billingsley, *Convergence of probability measures*, Wiley-Interscience publication, 1999.

15.070J / 6.265J Advanced Stochastic Processes
Fall 2013