6.265-15.070 Advanced Stochastic Processes
Take home final exam.

Date distributed: December 15
Date due: December 18

100 points total

Problem 1 (15 points) Suppose X_i is an i.i.d. zero mean sequence of random variables with a finite everywhere moment generating function $M(\theta) = \mathbb{E}[\exp(\theta X_1)]$. Argue the existence and express the following large deviations limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{1\le i\ne j\le n} X_i X_j > n^2 z \right)$$

in terms of $M(\theta)$ and z.

Problem 2 (15 points) Establish the following identity directly from the definition of the Ito integral:

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

Hint: Think about the integral $\int_0^t B_s ds$ as a limit of the sums $\sum_i B_{t_{i+1}}(t_{i+1} - t_i)$.

Problem 3 (15 points) Given a stochastic process X_t, the so-called Stratonovich integral $\int_0^t X_s \circ dB_s$ of X_t with respect to a Brownian motion B_t is defined as an $\mathbb{L}_2$ limit of

$$\lim_n \sum_{0\le i\le n-1} X_{t_i^*}(B_{t_{i+1}} - B_{t_i}),$$

over sequence of partitions $\Pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, with resolution $\Delta(\Pi_n) \to 0$ as $n \to \infty$, when such a limit exists. Here $t_i^* = \frac{t_i + t_{i+1}}{2}$. Compute the Stratonovich integral $\int_0^t B_s \circ dB_s$ and compare it with the Ito integral $\int_0^t B_s dB_s$

Problem 4 (10 points) Given a sequence of real values $x_n \in \mathbb{R}, n \ge 1$, consider the associated sequence of δ probability measures $\mathbb{P}_n = \delta_{x_n}$. Show that $\mathbb{P}_n$ is tight if and only if the set $x_n, n \ge 1$ is bounded. Show that if $\mathbb{P}_n = \delta_{x_n}$ converges weakly to some measure $\mathbb{P}$, then there exists a limit $\lim_n x_n = x$ and furthermore $\mathbb{P} = \delta_x$.

Problem 5 (15 points) Recall from the probability theory that a sequence of random variables $X_n : \Omega \to \mathbb{R}$ is said to converge to X in distribution if $F_{X_n}(x) \to F_X(x)$ for every point x, such that F is continuous at x. Here F_n and F denote the cumulative distribution functions of X_n and X respectively.

(a) Each random variables $Z : \Omega \to \mathbb{R}$ induces a probability measure $\mathbb{P}_Z$ on $\mathbb{R}$ equipped with the Borel σ-field defined by $\mathbb{P}_Z(B) = \mathbb{P}(Z \in B)$. Thus from X_n and X we obtain sequences of probability measures $\mathbb{P}_n$ and $\mathbb{P}$ on $\mathbb{R}$. Show that $\mathbb{P}_n$ converges weakly to $\mathbb{P}$ ($\mathbb{P}_n \Rightarrow \mathbb{P}$) if and only if X_n converges to X in distribution. Namely, the two notions of convergence are identical for random variables.

Hint: for one direction you might find it useful to use Skorohod Representation Theorem (which you would need to find/recall on your own) and the relationship between the almost sure convergence and convergence in distribution.

(b) Suppose X_n is sequence of random variables which converges in distribution to X, all defined on the same probability space Ω. Suppose the sequence of random variables Y_n on Ω converges to zero in distribution. Establish that $X_n + Y_n$ converges weakly to X.

Problem 6 (15 points) Exercise 1 from Lecture 21. (Refer to the lecture note for the statement of the problem).

Problem 7 (15 points) Suppose $X_n, n \geq 1$ is an i.i.d. sequence with a zero mean and variance σ^2. Suppose $\theta > 0$ is a fixed constant. Compute the following triple-limit

$$\lim_{z\to\infty} z^{-1} \lim_{t\to\infty} \lim_{n\to\infty} \log \mathbb{P}\left(\max_{1\leq k\leq nt} \left(\frac{\sum_{1\leq i\leq k} X_i}{\sqrt{n}} - \theta \frac{k}{n} \right) \geq z \right).$$

15.070J / 6.265J Advanced Stochastic Processes
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