Class 7 Outline

Advanced Topics in Simulation:

1. **Variance Reduction**
   A. **Common Random Numbers**
   B. **Antithetic Variables**
   C. **Control Variates**

2. **Simulation-Based Optimization**

Why Variance Reduction?

- **Example: Rare Event Probability (Insurance, Risk Analysis, Reliability, Public Health, etc.)**
- **Simulation Model for Catastrophic Event C with $p < 1/1M$ (exact value unknown)**

→ **How many trials to estimate p with 1% accuracy?**

- **Remember:**

$$Y \ n \qquad z_{1- \ /2}\sqrt{\frac{S^2 \ n}{n}}$$

- **Here (UB – LB) / E[C] < 1% → $n > 2.6 \times 10^{11}$!!!**

Why Variance Reduction?

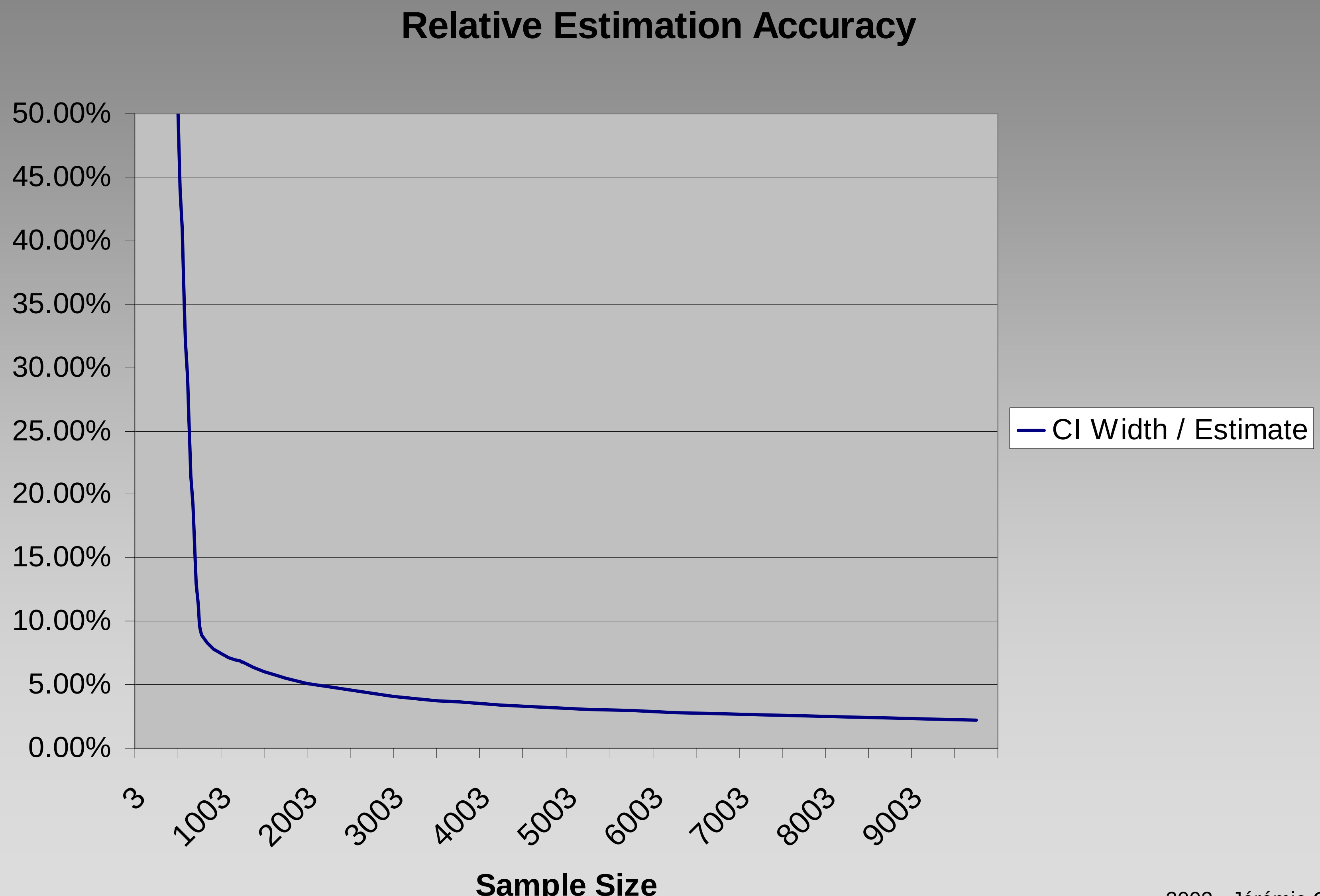
- **Cost estimate within 1% for policy RCNC2 in Ontario Gateway → 200,000 iterations !!!**

- **Instead of brute-force sample size approach, let's try to reduce the estimator variance…**

$$Y\ n \qquad z_{1-\ /2}\sqrt{\frac{S^2\ n}{n}}$$

Idea:
Reduce
this term!

The Wall



2002 - Jérémie Gallien

Common Random Numbers

- **This technique (CRN) is used when comparing alternatives**
- **Intuition?**
- **Implication: Use the same (synchronized) seeds of random numbers during simulation runs intended to compare alternatives**
- **Is this always beneficial?**

Formal CRN Argument

- **Want to estimate E[g(X)-h(X)]**
- **Generate $X_1, X_2, X_3, \ldots, X_n$, $Z_i = g(X_i)-h(X_i)$**
- **Estimator $Z(n) = (Z_1+Z_2+Z_3+ \ldots+Z_n) / n$**

$$\mathrm{Var}(Z(n)) = \mathrm{Var}(\, g(X)-h(X)\,) / n$$

$$= \Big(\mathrm{Var}\, g(X) + \mathrm{Var}\, h(X) - 2\mathrm{Cov}[g(X),h(X)] \Big) / n$$

- **CRN is only a good idea when g(X) and h(X) are positively correlated!**

Antithetic Variables (AV)

- **Idea: Take the average of negatively correlated (unbiased) estimators!**

- **What's the intuition?**

AV Implementation

- **Want to estimate E[h(X)]. Let $U_1, U_2, \ldots, U_n$ the Uniform[0,1] numbers used to generate $X_1, \ldots, X_n$**
- **Idea: Compute X'_i using $1 - U_i$!!!**

➡ **Why do X'_i and X_i have the same distribution?**

- **Estimators:** $Z(n) = (Z_1+Z_2+Z_3+ \ldots +Z_n) / n$

  $Z'(n) = (Z'_1+Z'_2+Z'_3+ \ldots +Z'_n) / n$

  with $Z_i = h(X_i)$ and $Z'_i = h(X'_i)$

- **Take $W(n) = (Z(n) + Z'(n)) / 2$!**

➡ **When does it work best / worst?**

Control Variates

- **Let Y be the raw output variable**
  Let X be some variable correlated to Y

- **Definition:**

$$Z = Y + c(X - E[X])$$

- **Intuition?**

- **Var Z = Var Y + c^2 Var X + 2c Cov(X,Y), so**
  Var Z is minimized when c = - Cov(X,Y) / Var X

Variance Reduction: Results

- **For the reliability example, the results obtained where:**

		Estimator		
		Raw	Antithetic	Control Variate
Standard Deviation	Absolute	0.144	0.100	0.023
	Relative	100%	69%	16%

Simulation-Based Optimization

- **Remember Monte-Carlo framework:**

Estimate $E[\, h(\mathbf{X})\,]$

where $\mathbf{X} = \{X_1,\ldots,X_m\}$ is a random vector in R^m

- **The associated optimization problem is:**

$\text{Max}_{\mathbf{d}}$ $E[\, h(\mathbf{X},\mathbf{q})\,]$

s.t. $\mathbf{q} \in \mathbf{F}$

where **X** is a random vector in R^m

q a vector of decision variables in R^w

F is a subset of R^w (feasible region)

Newsvendor Model

- **One time decision under uncertainty**
- **Trade-off: Ordering or producing**
  - **too much (waste, salvage value < cost) versus**
  - **too little (excess demand is lost)**

- **Examples:**
  - **Restaurant;**
  - **Fashion;**
  - **High Tech;**
  - **Capacity and Inventory decisions…**

Newsvendor Model Parameters

- **q = Order Quantity** — ***decision***

- **c = Unit Cost**
- **r = Unit Revenue**
- **s = Unit Salvage Value**

} ***parameters*** **(r > c > s)**

- **d = Demand (unknown)** — ***random variable***

Model Derivation

	• IF d > q (demand > order qty)	• IF d < q (demand < order qty)
Profit:	$q \cdot (r - c)$	$d \cdot (r - c) + (q - d) \cdot (s - c)$
Incremental Analysis:	$q \to q + 1$:	
Δ Profit:	*r - c*	*s - c*
EAP:	$P(d > q) \cdot (r\text{-}c) + P(d \le q) \cdot (s\text{-}c)$	

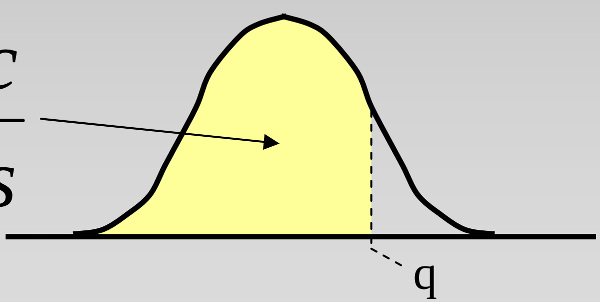
As long as the *Expected Additional Profit* [EAP] is positive, it is lucrative to increase *q* to *q + 1 !!!*

Newsvendor Formula

- **Set EAP = 0 to find:**

$$P(d < q) = \frac{r-c}{r-s} = \frac{r-c}{\underbrace{(r-c)}_{\text{cost of under-stocking}} + \underbrace{(c-s)}_{\text{cost of over-stocking}}} = \frac{k_u}{k_u + k_o}$$

$$\frac{r-c}{r-s}$$



Demand Distribution

Module Wrap-Up

- ***Class 1*** **Definitions, The Simulation Process**
- ***Class 2*** **Monte-Carlo Theory and Applications, Crystal Ball**
- ***Class 3*** **Ontario Gateway: Monte-Carlo Case**
- ***Class 4*** **Discrete-Event Theory and Applications, Simul8**
- ***Class 5*** **Human Genome Project: Discrete-Event Case**
- ***Class 6*** **Design of Simulation Experiments**
- ***Class 7*** **Advanced Topics: Variance Reduction, Simulation-Based Optimization**

Follow-up Classes

- **ESD.76J / 1.019 Systems Simulation**
  Spring, 12 credits. Comparable footprint.

- **1.021J Introduction to Modeling and Simulation**
  Spring, 12 credits. Continuous models, engineering applications.

- **2.141 Modeling and Simulation of Dynamics Systems**
  Fall, 12 credits. Advanced engineering simulation class.

Module Wrap-Up

Industrial decision-making is interdisciplinary: