

Using Laws of Probability



Sloan Fellows/Management of Technology
Summer 2003

Outline

- Uncertain events
- The laws of probability
- Random variables (discrete and continuous)
- Probability distribution
- Histogram (pictorial representation)
- Mean, variance, standard deviation
- Binomial distribution

Modeling Uncertain Events

- Uncertain (probabilistic) events are modeled as if they were precisely defined *experiments* with more than one possible *outcome*. Each outcome (or elementary event) is clear and well-defined, but we cannot predict with certainty which will occur.
- Note – “ambiguity” and “ignorance” refer to not knowing the exact probabilities of outcomes and not knowing all the possible outcomes, respectively.
- For example, the experiment “flipping 3 unbiased coins in order and recording their values (H for heads and T for tails)” has as one possible outcome:

$$H_1 T_2 T_3$$

Sample Space Events

- The set of all possible outcomes for an experiment is called the *Sample Space* for the experiment (labeled set S).
- The set of all possible outcomes for an experiment (Sample Space: S) is *mutually exclusive* (i.e. outcomes do not overlap) and *collectively exhaustive* (i.e. comprises all possible outcomes for the experiment)
 - We did not consider a coin landing on its edge

Events

- An *event* (e.g. event A) is a well-defined collection of outcomes for a particular experiment.
- We say that event A happens whenever one of the outcomes in event A happens (sounds silly, but “rainy days” are days when it rains).
- For example, in our experiment, if we define event A as “getting a total of 1 or fewer heads,” then, whenever any of the following outcomes happens, we say that event A happened:

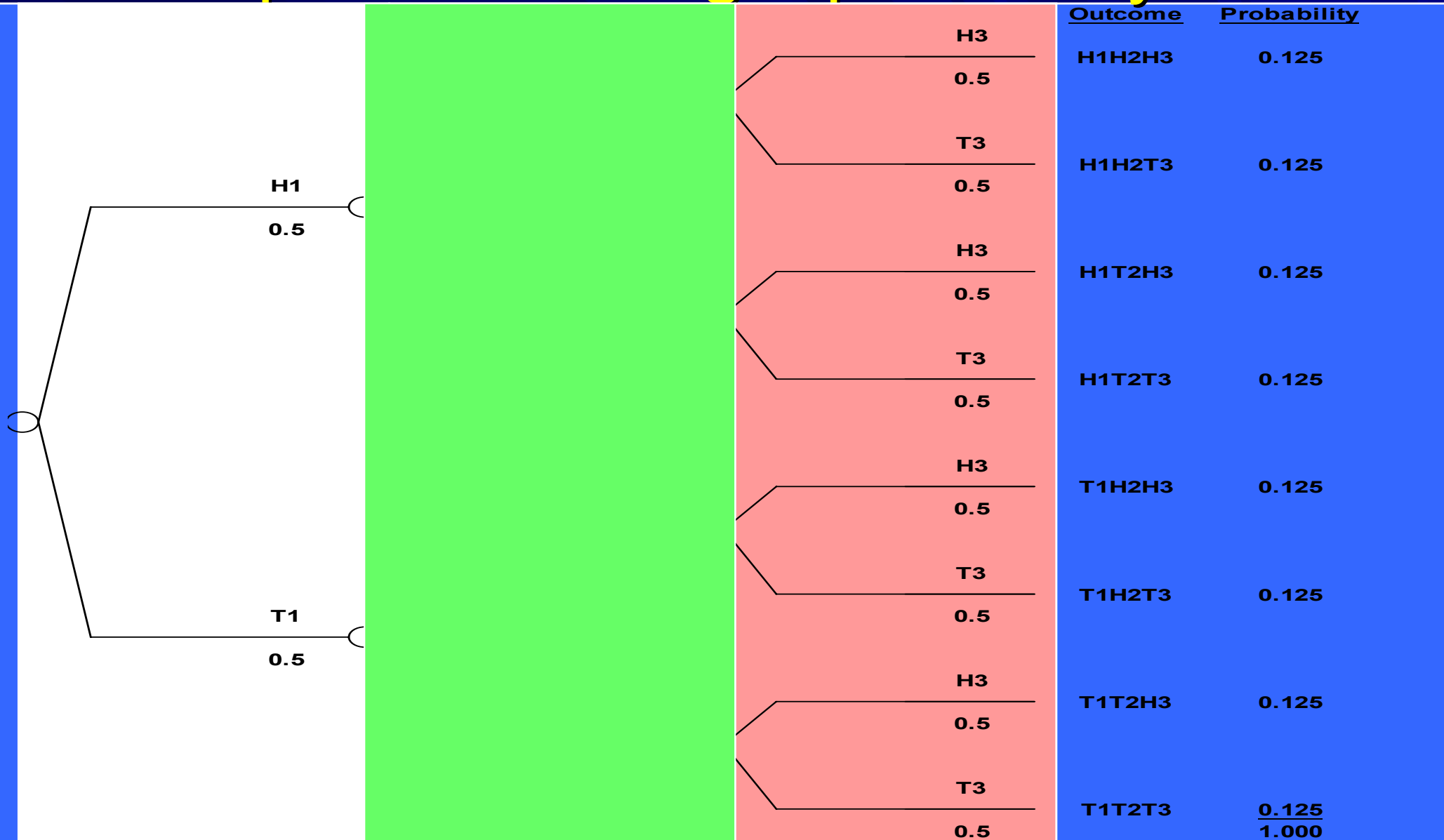
$T_1T_2T_3$

$H_1T_2T_3$

$T_1H_2T_3$

$T_1T_2H_3$

We can enumerate the sample space for our experiment using a probability tree:



First coin

Second coin

Third coin

Calculating probabilities

- Notice that since the coins are unbiased (i.e. $p(H)=1/2$) all outcomes have the same probability:

$$p(\text{any outcome}) = 1/(\text{total \# of outcomes})=1/8$$

- We can also calculate the probability for each outcome by multiplying the probabilities on the branches (this would allow us to easily calculate outcome probabilities for biased coins).
- Notice that, since the outcomes are collectively exhaustive, total probability adds up to 1.00
- In general, $0 \leq p(\text{any event}) \leq 1.00$
(*First Law of Probability*)

Intuitive Probability

- Which coin toss sequence is most likely?
Which is next most likely? (Q#11)
 1. HTHTTH
 2. HHHHTH **23% correct**
 3. HHHHTH
- Our intuitions are based on the pattern or “look” of randomness
- Actually calculating probability helps us to be explicit about our assumptions and to test them against reality

The Gambler's Fallacy

- We expect that probability is a self-correcting process, such that a “run” of one type of outcomes will be followed by a “run” of the opposite to “even things out”
- You have started buying stocks on the Internet, beginning with five different stocks. Each stock goes down soon after your purchase. As you prepare to make a sixth purchase, you reason that it should be more successful, since the last five were “lemons.” After all, the odds favor making at least one successful pick in six decisions. (Q #12) **93% got this right!**

The Second Law of Probability

- If A and B are *disjoint* (mutually exclusive) events:

$$p(A \text{ or } B) = p(A \cup B) = p(A) + p(B)$$

- The “fool proof” (but often brute force) way to calculate the probability of an event is by adding up the probabilities of all the elementary outcomes in the event.
- Recall event A: one or fewer heads in three tries
$$p(A) = p(T_1 T_2 T_3) + p(H_1 T_2 T_3) + p(T_1 H_2 T_3) + p(T_1 T_2 H_3)$$
$$= 4(1/8) = 1/2 \text{ (since all outcomes are equally likely)}$$

The General Second Law

- We also have (not in the book) that the *general form* of the second law (to avoid double counting overlapping events) is

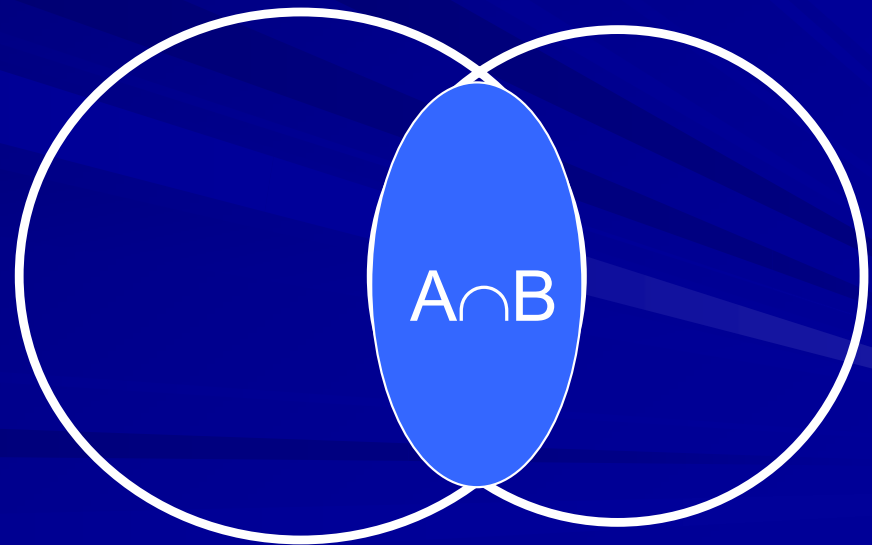
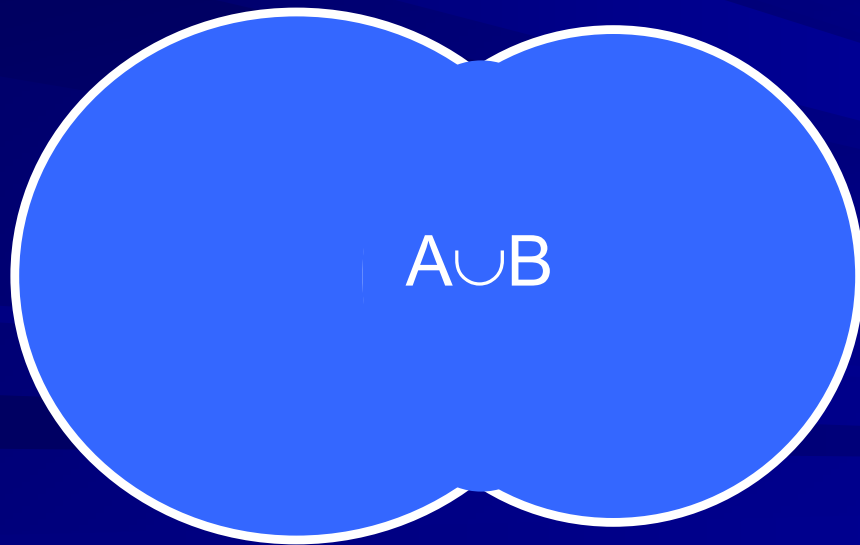
$$p(A \text{ or } B) = p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

Example:

- Event A ($\# H \leq 1$): $\{H_1 T_2 T_3, T_1 H_2 T_3, T_1 T_2 H_3, T_1 T_2 T_3\}$
- Event B ($\# H \geq 1$): $\{H_1 T_2 T_3, T_1 H_2 T_3, T_1 T_2 H_3, H_1 H_2 H_3, H_1 H_2 T_3, H_1 T_2 H_3, T_1 H_2 H_3\}$
- $p(A \text{ or } B) = p(A \cup B) = p(A) + p(B) - p(A \cap B) =$
 $= 4/8 + 7/8 - 3/8 = 8/8 = 1.0$ (as expected!)

Notation

- $A \cup B$ is “A or B” and the union of A and B
- $A \cap B$ is “A and B” and the intersection of A and B
- I remember them as a boat with oars (“or”) vs. an ant hill (“and” hill, sorry)



The Third Law of Probability

- For events A and B the probability that event A occurred given that B occurred:

$$p(A/B) = p(A \cap B) / p(B); \text{ or } p(A \cap B) = p(A/B)p(B)$$

$$p(A/B) = (3/8) / (7/8) = 3/7.$$

(Check: If we know B happened, we know we are “focusing” on 7 events. Out of these 7 events, 3 belong to event A: 1 head or fewer.)

Fourth Law of Probability

- If A and B are **independent** events then:

$$p(A/B) = p(A) \quad (1)$$

(knowing that B happened does not change the likelihood of A)

- Using the 3rd Law (conditioning on B) and (1):

$$p(A \cap B) = p(A/B)p(B) = p(A) p(B) \quad (2)$$

- From the 3rd Law (but conditioning on A):

$$p(A \cap B) = p(B/A)p(A) \quad (3)$$

- Since (3) & (2) are equivalent expressions for $p(A \cap B)$:

$$p(A/B)p(B) = p(B/A)p(A) = p(A) p(B) \quad (4)$$

Thus, $p(B/A) = p(B)$ (5)

If A is independent of B; then B is independent of A.

Probability Tables

- 100 mutual funds, 40 Rose and 60 Fell; from 3 fund companies A, B, C (convert into probabilities by dividing each quantity by 100).

- Events (example): Probability that a randomly selected fund of these 100 rose and was from company A:

$$p(A \cap R) = 0.3$$

	Fell (F)	Rose (R)	<i>Total</i>
Firm (A)	$p(A \cap F)$ 4	$p(A \cap R)$ 30	$p(A)$ 34
Firm (B)	$p(B \cap F)$ 6	$p(B \cap R)$ 15	$p(B)$ 21
Firm (C)	$p(C \cap F)$ 30	$p(C \cap R)$ 15	$p(C)$ 45
<i>Total</i>	$p(F)$ 40	$p(R)$ 60	100

- Events are disjoint so we can add probabilities across rows and columns:
 $p(A) = p(A \cap F) + p(A \cap R) = 0.34$
 $p(R) = p(A \cap R) + p(B \cap R) + p(C \cap R) = 0.60$

Conditional Probabilities

- To calculate *conditional probabilities* we *focus* on the conditioning event's column/row.
Example:

- The probability that the fund selected fell (F) given that it is from company C:

$$p(F/C) = p(C \cap F) / p(C) \\ = .30 / .45 = 2/3;$$

Also,

$$p(R/C) = p(C \cap R) / p(C) \\ = .15 / .45 = 1/3$$

	Fell (F)	Rose (R)	Total
Firm (A)	$p(A \cap F)$ 4	$p(A \cap R)$ 30	$p(A)$ 34
Firm (B)	$p(B \cap F)$ 6	$p(B \cap R)$ 15	$p(B)$ 21
Firm (C)	$p(C \cap F)$ 30	$p(C \cap R)$ 15	$p(C)$ 45
Total	$p(W)$ 40	$p(R)$ 60	100

- Notice these conditional probabilities are for disjoint events so we can add them:
 $p(F/C) + p(R/C) = 2/3 + 1/3 = 1.00$
(a fund from company C has to either fall or rise!)

Does Statistical Information Make Sense By Itself ?

- A cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city. **85% of the cabs in the city are Green; 15% are Blue.**

A witness identified the cab as a Blue cab. When presented with a sample of cabs (half of which were Blue, half Green), the witness was correct 80%.

52% said 80%; 12% said 12% or 15%

- What is the probability that the cab involved in the accident was Blue?

Statistical Information, cont.

- A cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city. **Although the two companies are roughly equal in size, 85% of the cab accidents in the city involve Green cabs; 15% involve Blue cabs.**

A witness identified the cab as a Blue cab. When presented with a sample of cabs (half of which were Blue, half Green), the witness was correct 80%. 36% said 80%; 46% said 12% or 15%

- What is the probability that the cab involved in the accident was Blue?

Statistical Information, cont.

- Medians are 80% in first case; 60% in the second case

Cab is:

	Green □	Blue □	
Witness says: Green □	68 □	3 □	71 □
Blue □	17 □	12 □	29 □
	85 □	15 □	100 □

- Probability = $12/29 = 41\%$

Bayes' Theorem

$$p(B/W_b) = p(W_b/B) \times p(B) / p(W_b)$$

$$p(W_b) = p(W_b/B) \times p(B) + p(W_b/G) \times p(G)$$

$$p(B) = .15$$

$$p(W_b/B) = .80$$

$$p(G) = .85$$

$$p(W_b/G) = .20$$

$$p(W_b) = .80 \times .15 + .20 \times .85 = .29$$

$$p(B/W_b) = .80 \times .15 / .29 = .41$$

Summary and Next Session

- The Laws of Probability help us to fill out probability tables and create decision trees
- Intuitive judgment about probability is often incorrect
- For Monday's session, you must hand in the Graphics Corp. case by the start of class. Remember, up to two pages of text plus up to six pages of supporting materials.