Optimization Methods in Management Science

MIT 15.053

RECITATION 6
TAS: GIACOMO NANNICINI, EBRAHIM NASRABADI

Problem 1

The purpose of this recitation is to familiarize students with a variety of integer programming modeling techniques as described in the IP Formulation Guide and in the powerpoint tutorial on IP formulations.

We start with an integer program IP1 defined as follows:

$$\left.\begin{array}{rrcl}
\max & 21x_1 + 32x_2 + 40x_3 + 49x_4 + 57x_5 + & & \\
 & +71x_6 + 82x_7 + 91x_8 + 100x_9 + 109x_{10} & & \\
\text{s.t.:} & 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 + & & \\
 & +7x_6 + 8x_7 + 9x_8 + 10x_9 + 11x_{10} & \leq & 900 \\
\forall i = 1, \ldots, 3 & x_i & \in & \{0,1\} \\
\forall i = 4, \ldots, 10 & 0 \leq x_i & \leq & 100.
\end{array}\right\} \text{(IP1)}$$

For each of the parts below, you are to add constraint(s) and possibly variables to ensure that the logical condition is satisfied by the integer program. Each part is independent; that is, no part depends on the parts preceding it. You do not need to repeat the integer programming objective or constraints given above. You may use the big M method for formulating constraint when it is appropriate.

(a) (4 points) Write a single linear constraint that is equivalent to the statement "If $x_1 = 1$, then $x_2 = 0$."

Solution. $x_1 + x_2 \leq 1$

(b) (4 points) Write a single linear constraint that is equivalent to the statement "$x_2 = 1$ or $x_3 = 0$," but not both.

Solution. $x_2 = x_3$

(c) (4 points) Add a binary variable w_1, and add two constraints that ensure that $w_1 = 1$ if $x_5 + x_6 \geq 70$, and $w_1 = 0$ if $x_5 + x_6 \leq 69$.

Solution. $x_5 + x_6 \geq 70 - M(1 - w_1)$ and $x_5 + x_6 \leq 69 + Mw_1$.

(d) (4 points) Add 3 binary variables w_2, w_3, and w_4 and at most 4 constraints so as to ensure that at least one of the following constraints is satisfied: (i) $x_5 \leq 92$, (ii) $x_6 \leq 40$, (iii) $x_7 + x_8 \geq 74$.

Solution. $x_5 \leq 92 + M(1 - w_2)$, $x_6 \leq 40 + M(1 - w_3)$, $x_7 + x_8 \geq 74 - M(1 - w_4)$ and $w_2 + w_3 + w_4 \geq 1$.

(e) (4 points) Add a single binary variable w_5 and two constraints to ensure that at least one of the following two constraints are satisfied (i) $x_9 \leq 45$, (ii) $x_{10} \geq 22$.

Solution. $x_9 \leq 45 + Mw_5$ and $x_{10} \geq 22 - M(1 - w_5)$.

(f) (4 points) Add a single integer variable w_6 and a constraint that ensures that x_8 is divisible by 2 but not divisible by 4. (The remainder when dividing by 4 must be 2).

Solution. $x_8 - 4w_6 = 2$.

(g) (4 points) Add three binary variables w_7, w_8, and w_9 and two constraints that ensures that $x_{10} = 13$ or 39 or 88.

Solution. $x_{10} = 13w_7 + 39w_8 + 88w_9$ and $w_7 + w_8 + w_9 = 1$.

(h) (4 points) Add variable(s) and constraint(s) that model the cost of x_4 as $f_4(x_4)$, which is defined as follows: If $x_4 = 0$, then $f_4(x_4) = 0$. If $x_4 \geq 1$, then $f_4(x_4) = 250 + 49x_4$.

Solution. We add a binary variable w_{10} and model the cost of x_4 as: $250w_{10} + 49x_4$, subject to the constraints: $w_{10} \leq x_4 \leq 100w_{10}$.

(i) (8 points) Add variable(s) and constraint(s) that model the cost of x_5 as $f_5(x_5)$, which is defined as follows: If $0 \leq x_5 \leq 10$, then $f_5(x_5) = 57x_5$. If $11 \leq x_5 \leq 20$, then $f_5(x_5) = 570$. If $21 \leq x_5 \leq 100$, then $f_5(x_5) = -480 + 50x_5$.

Solution.

$$\left.\begin{array}{rrcl}
\text{Obj function:} & 57y_1 + 570w_2 - 480w_3 + 50y_3 & & \\
\text{s.t.:} & w_1 + w_2 + w_3 & = & 1 \\
 & y_1 + y_2 + y_3 & = & x_5 \\
 & 0w_1 \leq y_1 & \leq & 10w_1 \\
 & 11w_2 \leq y_2 & \leq & 20w_2 \\
 & 21w_3 \leq y_3 & \leq & 100w_3 \\
 & w_1, w_2, w_3 & \in & \{0, 1\}.
\end{array}\right\}$$

Problem 2

As the leader of an oil-exploration drilling venture, you need to determine which 5 sites out of 10 to evaluate for drilling opportunities. The goal is to select 5 sites with the lowest overall cost. Label the sites $S_1, S_2, \ldots, S_{10}$, and the exploration costs associated with each as $c_1, c_2, \ldots, c_{10}$. Regional development restrictions are such that:

(i) Evaluating sites S_2 and S_7 will prevent you from evaluating either site S_6 or S_9.

(ii) Evaluating sites S_1 and S_3 will prevent you from also evaluating both sites S_5 and S_6.

(iii) Evaluating site S_3 or S_4 prevents you from evaluating site S_6.

(iv) Of the group S_3, S_6, S_7, S_8, at most two sites may be assessed.

Formulate an integer program to determine the minimum-cost exploration scheme that satisfies these restrictions. Try to develop a model in which the only variables are $x_1, \ldots, x_{10}$, where x_j is 1 or 0 according as site j is evaluated or not. (For example, the constraint "Evaluating sites S_2 and S_7 will prevent you from exploring site S_6" can be expressed as $x_2 + x_6 + x_7 \leq 2$ because the only binary solutions prohibited have $x_2 = x_6 = x_7 = 1$.)

Solution. The formulation depends on how you interpreted the constraint logic. In this case, well let $x_i = 1$ if we drill at site i, and 0 otherwise. For all parts our objective function is as

$$\min \sum_{i=1}^{10} c_i x_i.$$

Our only constraint initially is that we must select 5 sites, thus:

$$\sum_{i=1}^{10} c_i x_i = 5.$$

The additional constraints for each part are as follows:

(i) For this condition, we need two constraints:

$$x_2 + x_7 + x_6 \leq 2,$$
$$x_2 + x_7 + x_9 \leq 2.$$

Each constraint restricts the system to only explore two out of the three sites listed. Thus, if x_2 and x_7 both equal 1, x_6 and x_9 must both be 0.

(ii) We can still represent this condition with one constraint as follows:

$$x_1 + x_3 + x_5 + x_6 \leq 3.$$

Essentially, we can select up to 3 of the set, but we any 4^{th} selection would break the constraint as expected.

(iii) We can represent this with two constraints:

$$x_3 + x_6 \leq 1,$$
$$x_4 + x_6 \leq 1.$$

(iv) This type of constraint is the easiest to translate as

$$x_3 + x_6 + x_7 + x_8 < 2.$$

Problem 3

Suppose you want to minimize or maximize a piecewise linear function of one variable, subject to linear constraints. This is a problem that can be solved by resorting to linear constraints only, possibly by adding extra variables. In this example, we consider the function with three pieces shown in Figure 1.

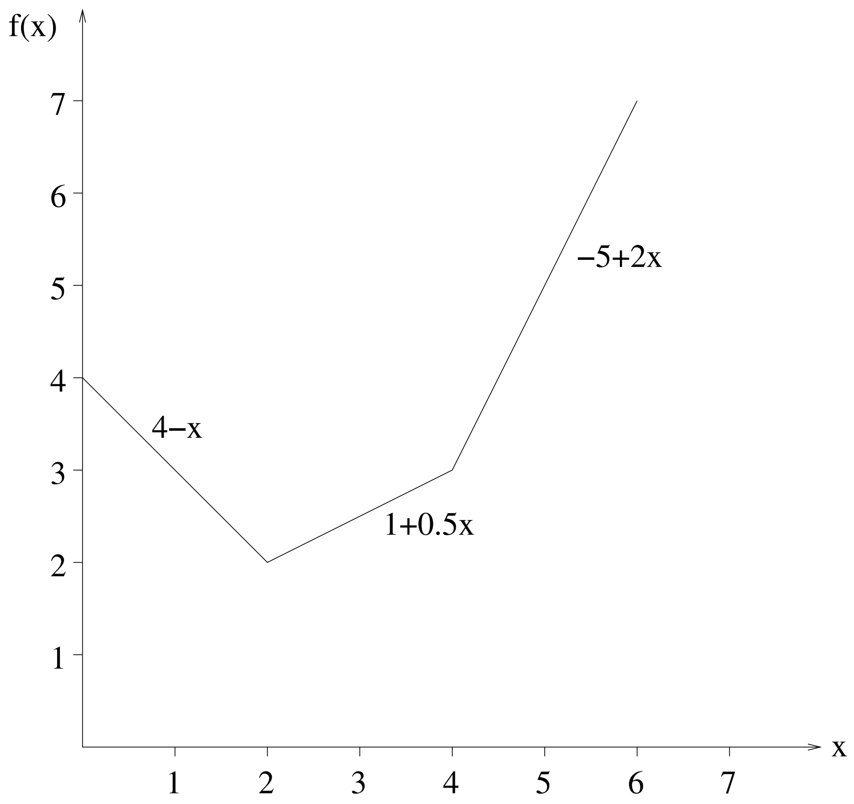

Figure 1: Piecewise linear function discussed in Problem 5.

(a) Suppose we want to minimize $f(x)$ shown in Figure 1. Assume that x is subject to a set of linear constraints that involve other variables $A(x'|x) = b$, so that we cannot simply solve the problem by inspection because we do not know what values x will take. How can we formulate this problem in linear form? Do we need integer variables? (In your formulation, you can ignore the additional constraints $A(x'|x) = b$.)

Solution. Notice that $f(x)$ can be expressed as the maximum of the three linear functions, that is,

$$f(x) = \max\{4 - x, 1 + 0.5x, -5 + 2x\}$$

So the problem is to formulate

$$\max f(x) = \min \ \max\{4 - x, 1 + 0.5x, -5 + 2x\}.$$

This problem can be formulated as a linear program by introducing an extra variable y and setting $y = \max\{4 - x, 1 + 0.5x, -5 + 2x\}$. This implies that y is greater that or equal to each piece of $f(x)$. This results in the following formulation:

$$\left.\begin{array}{lrcl} \min & y & & \\ \text{s.t.:} & 4 - x & \leq & y \\ & 1 + 0.5x & \leq & y \\ & -5 + 2x & \leq & y \\ & x, y & \geq & 0. \end{array}\right\}$$

(b) Consider now the problem of maximizing $f(x)$ of Figure 1, subject to a set of linear constraints that involve other variables. We cannot use the same approach of Part 2.A. Explain why and find an alternative way of formulating the problem, adding (binary or integer) variables as needed.

Solution. As mentioned previously, $f(x)$ is the maximum of a number of linear functions. Therefore, we deal with the following situation

$$\max f(x) = \max \ \max\{4 - x, 1 + 0.5x, -5 + 2x\}.$$

In general, such a problem cannot be expressed in the simple way of Part 5.A. We must use a more general approach where we select the active piece of the piecewise linear function with binary variables, then maximize the corresponding piece.
We introduce two new variables for every pice of the function:

$$w_1 = \begin{cases} 1 & \text{if } 0 \le x < 3, \\ 0 & \text{otherwise.} \end{cases} \qquad x_1 = \begin{cases} x & \text{if } 0 \le x < 3, \\ 0 & \text{otherwise.} \end{cases}$$
$$w_2 = \begin{cases} 1 & \text{if } 2 \le x < 4, \\ 0 & \text{otherwise.} \end{cases} \qquad x_2 = \begin{cases} x & \text{if } 2 \le x < 4, \\ 0 & \text{otherwise.} \end{cases}$$
$$w_3 = \begin{cases} 1 & \text{if } 4 \le x \le 6, \\ 0 & \text{otherwise.} \end{cases} \qquad x_3 = \begin{cases} x & \text{if } 4 \le x \le 6, \\ 0 & \text{otherwise.} \end{cases}$$

We can now write the integer program as follows:

$$\left.\begin{array}{lrcl} \max & 4w_1 - x_1 + 1w_2 + 0.5x_2 - 5w_3 + 2x_3 & & \\ \text{s.t.:} & x_1 + x_2 + x_3 & = & x \\ & w_1 + w_2 + w_3 & = & 1 \\ & 0w_1 \le x_1 & \le & 2w_1 \\ & 2w_2 \le x_2 & \le & 4w_2 \\ & 4w_3 \le x_3 & \le & 6w_3 \\ & w_1, w_2, w_3 & \in & \{0,1\}. \end{array}\right\}$$

We next show that each solution of the above integer program corresponds to a point of the original problem with the same objective function value and vice versa. Suppose that $(\mathbf{w}, \mathbf{x}) = (\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x})$ is a feasible solution to the integer program. Then exactly one of the three binary variables w_1, w_2, w_3 must be 1. Suppose that $w_2 = 2$. It then follows from the constrains that $w_1 = w_3 = 0$, and in addition $2 \le x_2 = x \le 4$. Moreover, the objective function value for $(\mathbf{w}, \mathbf{x})$ is $1 + 0.5x$. Hence, the solution $(\mathbf{w}, \mathbf{x})$ corresponds to the point x $(2 \le x \le 4)$ with the same the objective function value.
Now consider a point x in [0,6]. Assume that $2 \le x \le 4$. In this case, by setting $w_2 = 1, w_1 = w_3 = 0, x_1 = x_3 = 0, x_2 = x$, we get a feasible solution for the integer program, whose objective function value equals $1 + 0.5x$.
In Problem 4, the function $f(x)$ is the maximum of a number of linear functions. Therefore, f is a convex piecewise linear function and minimizing such a function can be represented by a linear program, but we need an integer program to formulate $\min f(x)$. On the other hand, minimizing a concave piecewise linear function (a function that can be

written as the minimum of a number of linear functions) can be expressed by a linear program, but we need an integer program to formulate the maximization version.

15.053 Optimization Methods in Management Science
Spring 2013