Optimization Methods in Management Science

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Recitation 2

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Problem 1

This problem verifies your understanding of the geometry of Linear Programming. The questions are not trivial, therefore consider carefully your answer.

Consider the feasible region depicted in Figure 2 (note that it extends indefinitely towards the upper right part of the graph). Here is a proposition that is valid for all linear programs and will be useful for solving this problem. For any point p, let $c(p)$ be the cost of point p.
Proposition. If point p' is on the line segment joining points p and p'', then:

- $c(p') \geq \min\{c(p), c(p'')\}$, and
- $c(p') \leq \max\{c(p), c(p'')\}$.

For example, in Figure 2 we have $c(E) \geq \min\{c(D), c(F)\}$, and $c(E) \leq \max\{c(D), c(F)\}$. This follows from simple linear algebra: if p' is on the line segment between p and p'', then

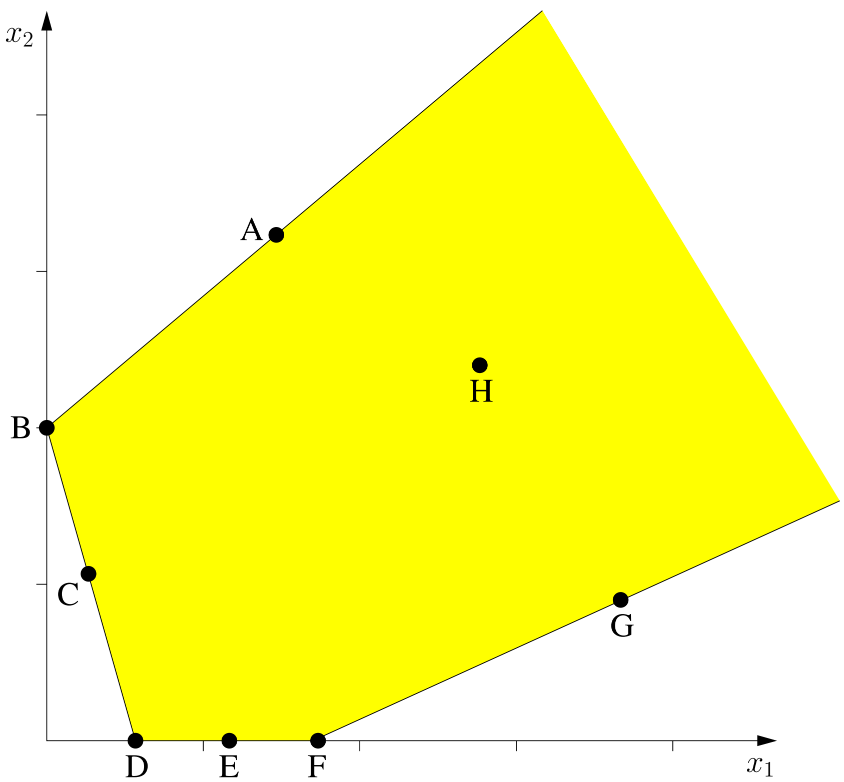

Figure 1: Feasible region discussed in Problem 3.

$p' = \lambda p + (1-\lambda)p''$ for some $0 \leq \lambda \leq 1$. The cost of a point is a linear fuction and therefore can be expressed as $c^\top p$. Then it is straightforward to verify that $c^\top p' = \lambda c^\top p + (1-\lambda)c^\top p'' \geq \lambda \min\{c(p), c(p'')\} + (1-\lambda)\min\{c(p), c(p'')\} = \min\{c(p), c(p'')\}$. The proof for the second statement is similar.

Questions. Answer the set of True/False questions below.

(a) F cannot be a unique optimum of the problem.

(b) If C is an optimal solution, D is also optimal.

(c) If A and B are optimal, the problem is unbounded.

(d) If B and F are optimal, G is not optimal.

(e) If no point among B, D and F is optimal, the problem is unbounded.

(f) There exists an objective function such that the problem is infeasible.

(g) If B, D and F are not optimal, the problem is infeasible.

(h) D and F could simultaneously be the only optima of the problem.

(i) If D and G are optimal, there is an infinite number of feasible solutions.

(j) If the problem has a finite optimal objective value, G could be an optimal solution.

(k) There exists an objective function such that H is optimal but A is not.

(l) If H is an optimal solution, there are infinitely many optimal solutions and the limit of the objective function values is plus or minus infinity.

Problem 2

Consider the feasible region defined by the following constraints:

$$\left.\begin{array}{rcl} x_1 - x_2 & \geq & -1.5 \\ x_1 - 2x_2 & \leq & 2 \\ 4x_1 + x_2 & \geq & 2 \\ x_1, x_2 & \geq & 0. \end{array}\right\} \qquad \text{(LP2)}$$

(a) Draw the feasible region of (LP2). Does this LP have an optimal solution for all possible objective functions? Why?

(b) Give an example of:

- An objective function such that both $(0,0)$ and $(0,1.5)$ are optimal (in minimization form)
- An objective function such that only the point $(0,1.5)$ is optimal (in minimization form).
- An objective function such that (LP2) is unbounded (in maximization form).

If such an example does not exist, explain why.

Problem 3

Consider the following LP:

$$\left.\begin{array}{lrcl} \max & 10x_1 + 8x_2 - 3x_3 & & \\ \text{s.t.:} & 2x_1 + 4x_2 - 0.5x_3 & \leq & 6 \\ & -2x_1 + 6x_2 - 4.5x_3 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \\ & x_3 & & \text{free.} \end{array}\right\} \quad \text{(LP3)}$$

(a) Write (LP3) in canonical form. If you have to introduce extra variables, explain what they stand for. Compute the initial basic feasible solution and write its value for *all* of the problem's variables (regardless of whether they are present in the original formulation or introduced for the canonical form).

(b) Write the initial simplex tableau and perform two iterations of the simplex algorithm. Is the basic solution after two iterations optimal? Why?

(c) In (LP3), replace the first constraint $2x_1 + 4x_2 - 0.5x_3 \leq 6$ with $-2x_1 + 4x_2 - 0.5x_3 \leq 6$. Write the initial simplex tableau (note that only one coefficient changes with respect to the first tableau of Part 2.B). Perform one iteration of the simplex algorithm. What happens in this case?

Problem 4

Here we review the Simplex Algorithm in more detail, without an Excel spreadsheet to provide guidance. Be careful when carrying out the calculations.

Consider the following linear program:

$$\left.\begin{array}{lrcl} \max & 2x_1 + 4x_2 & & \\ \text{s.t.:} & & & \\ & 0.5x_1 - 5x_2 & \leq & 12 \\ & x_1 + 2x_2 & \geq & -2 \\ & x_2 + x_3 & \geq & 4 \\ & x_1, x_2, x_3 & \geq & 0. \end{array}\right\}$$

(a) Write the initial simplex tableau. To do so, you will have to transform the problem into canonical form. What is the initial basic feasible solution? (Hint: at least one of the original variables is basic)

(b) Perform one iteration of the simplex algorithm, by pivoting in the variable with the largest reduced cost. Write down the candidate variables for pivoting out, and the corresponding value of the ratio test. Finally, report the simplex tableau after the first iteration.

(c) Would the simplex algorithm terminate after the first iteration? Why? Can you guess the optimal objective function value?