Optimization Methods in Management Science

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Recitation 1
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Problem 1

You create your own start-up company that caters high-quality organic food directly to a number of customers. You receive a number of tentative orders and you now have to tell your customers which orders you are going to take. Before embarking on this journey, you first want to allocate your production capabilities in order to devise a feasible daily production plan that maximizes your profit.

There are only three different kinds of food that you offer at this early stage of the company: Hummus (H) with garlic pitas, an excellent Moussaka (M), and a traditional Tabouleh (T) with parsley and mint.

Each meal has to be cooked, packaged and delivered. Each operation is run by yourself. You have to deliver between 12PM and 2PM everyday, and the food is made on the same day, therefore you estimate that the total number of available cooking hours is 4, the total number of packaging hours is 2, and the total number of delivery hours is 2.

Cooking sufficient Hummus for 10 portions requires 1 hour of time, packaging is done at the rate of 20 portions per hour, and delivery at the rate of 30 per hour. The cost of the ingredients for 1 portion is \$1, and each packaged portion can be sold for \$7.

Moussaka takes more time to prepare: in one hour, the food cooking team can prepare 5 portions. Packaging is done at the rate of 15 portions per hour. Since the Moussaka has to be delivered while still warm out of the oven, it is delivered in smaller batches, therefore only 15 portions can be delivered in one hour. The cost of the ingredients for 1 portion is \$2, and it can be sold for \$12.

Finally, Tabouleh can be prepared at the rate of 15 portions per hour, it can be packaged at the rate of 25 portions per hour, and delivered at the rate of 30 per hour. Tabouleh is very inexpensive and one portion only costs \$0.5 in raw ingredients, and can be sold for \$5.

Customers expressed interest in having the following products delivered every day: 20 Hummus meals, 10 Moussaka meals, and 30 Tabouleh meals.

Part 1.A

What is the best combination of meals in order to maximize profit? We can assume that meals do not have to be produced in even numbers – that is, we allow a non-integer solution.

Write the corresponding Linear Programming formulation on paper, labeling each decision variable and constraint with a proper name (nonnegativity constraints need not be labeled, but do not forget to include them!). Then use Excel to solve the problem. (Hint: the number of Moussaka meals is between 5 and 10 in the optimal solution.)

Solution. We define the following decision variables:

- x_H : number of Hummus meals,
- x_M : number of Mussaka meals,

- x_T : number of Tabouleh meals.

The decision problem can be written as follows:

$$\left.\begin{array}{rrcl} \min & 6x_H + 10x_M + 4.5x_T & & \\ \text{Cooking:} & (1/10)x_H + (1/5)x_M + (1/15)x_T & \leq & 4 \\ \text{Packaging:} & (1/20)x_H + (1/15)x_M + (1/25)x_T & \leq & 2 \\ \text{Delivery:} & (1/30)x_H + (1/15)x_M + (1/30)x_T & \leq & 2 \\ \text{DemandH:} & x_H & \leq & 20 \\ \text{DemandM:} & x_M & \leq & 10 \\ \text{DemandT:} & x_T & \leq & 30 \\ & x_H, x_M, x_T & \geq & 0, \end{array}\right\} \quad \text{(LP)}$$

The corresponding optimal solution is: $x_H = 8, x_M = 6, x_T = 30$, with a corresponding total profit of \$243.

Part 1.B

Your little brother offers his help for one hour a day. You assume that he can work as fast as you do, and he can use his bike if needed for delivery, but he can only help with one of the three tasks: cooking, packaging or delivery (not all of them). He asks \$10 dollars as a compensation. Should you accept his help? In case of a positive answer, would it be better to ask him to help with cooking, packaging or delivery? (Hint: compute the change in profit if you increase cooking, packaging or delivery time availability by one hour.)

Solution. If we resolve problem **??** increasing by one the right-hand side value of the Cooking constraint we obtain a solution with objective value: \$257.5, yielding a profit increase of \$14.5 compared to our initial solution.

If we resolve problem **??** increasing by one the right-hand side value of the Packaging constraint we obtain a solution with objective value: \$255 yielding a profit increase of \$12 compared to our initial solution.

If we resolve problem **??** increasing by one the right-hand side value of the Packaging constraint we obtain a solution with objective value: \$243 yielding no profit increase compared to our initial solution.

Therefore, it is worth asking our little brother to help us with Cooking for one hour at the cost of \$10, because our revenue increases by more than the additional cost.

Part 1.C

There is a drop in the demand of Hummus: instead of 20 meals, only 10 are now requested.

Does this change the optimal combination of meals to maximize profit? Could you have guessed without using Excel to solve the new problem?

Solution. The optimal solution does not change. This could have been guessed by noticing that we produce $x_H = 8 \leq 10$ Hummus meals in the optimal allocation. Adding a new constraint that is satisfied by the current optimal solution does not affect the solution.

Part 1.D

Because you do not want to cook only one kind of meal over and over again, you decide that none of the foods should make up more than 50% of the total portions prepared. How can you add this requirement to the Linear Program defined in Part 1.A (Hint: you may need more than one constraint)? Is the resulting mathematical program still linear? If not, is there a way to write it in linear form?

Solution. Requiring that the number of Hummus meals does not exceed 50% of the total production can be written as: $x_H/(x_H + x_M + x_T) \leq 1/2$. Similarly, for Moussaka we can write: $x_M/(x_H + x_M + x_T) \leq 1/2$. And for Tabouleh: $x_T/(x_H + x_M + x_T) \leq 1/2$. However these constraints are not linear because we have decision variables at the denominator. We can transform them into linear constraints by multiplying through by $(x_H + x_M + x_T)$, which is nonnegative. Note that by doing this we are excluding the solution $x_H = 0, x_M = 0, x_T = 0$, but we can do this because we know that the zero solution is not optimal.

The final Linear Program becomes therefore:

$$\left.\begin{array}{rrcl}
\text{min} & 6x_H + 10x_M + 4.5x_T & & \\
\text{Cooking:} & (1/10)x_H + (1/5)x_M + (1/15)x_T & \leq & 4 \\
\text{Packaging:} & (1/20)x_H + (1/15)x_M + (1/25)x_T & \leq & 2 \\
\text{Delivery:} & (1/30)x_H + (1/15)x_M + (1/30)x_T & \leq & 2 \\
\text{DemandH:} & x_H & \leq & 20 \\
\text{DemandM:} & x_M & \leq & 10 \\
\text{DemandT:} & x_T & \leq & 30 \\
\text{MaxRatioH:} & (1/2)x_H - (1/2)x_M - (1/2)x_T & \leq & 0 \\
\text{MaxRatioM:} & -(1/2)x_H + (1/2)x_M - (1/2)x_T & \leq & 0 \\
\text{MaxRatioT:} & -(1/2)x_H - (1/2)x_M + (1/2)x_T & \leq & 0 \\
 & x_H, x_M, x_T & \geq & 0,
\end{array}\right\} \qquad (\text{LP}')$$

Problem 2

Consider the functions depicted in Figure 1.

Part 2.A

Which functions represented in Figure 1 are convex (select one or more)?

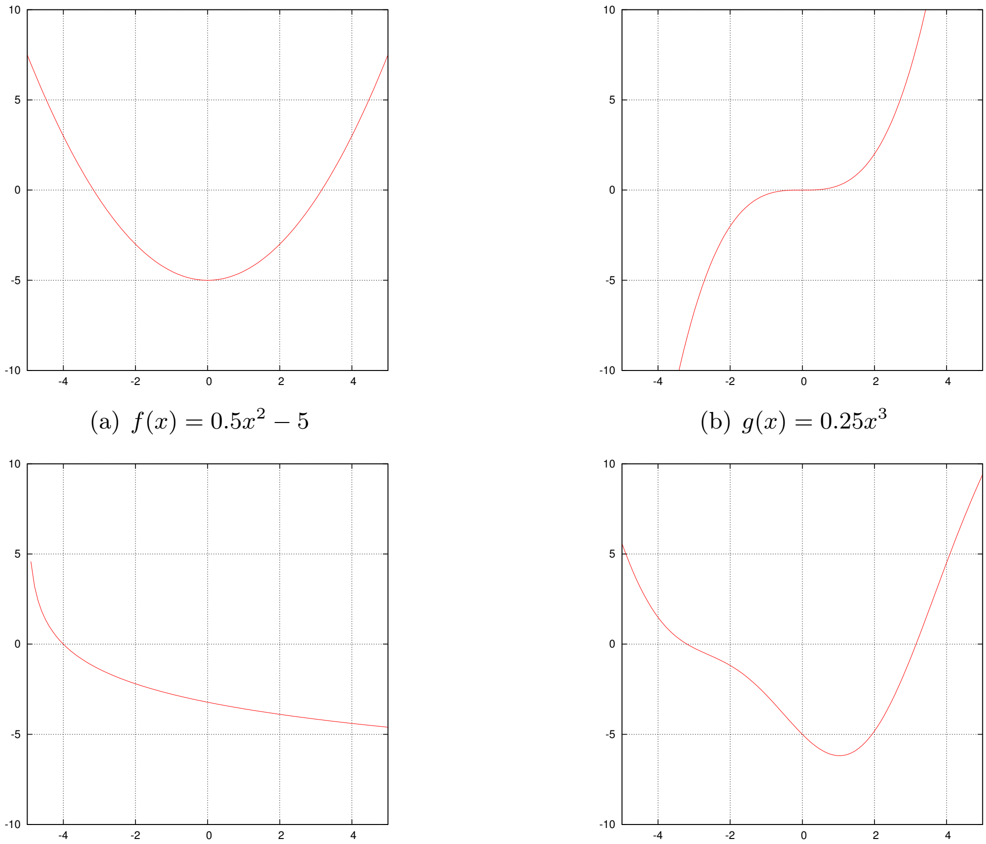
a) $f(x) = 0.5x^2 - 5$

b) $g(x) = 0.25x^3$

c) $h(x) = -2\log(x + 5)$

d) $\ell(x) = 0.5x^2 - 2\sin x - 5$

Solution. a) and c) are convex. b) and d) are not.



(a) $f(x) = 0.5x^2 - 5$

(b) $g(x) = 0.25x^3$

(c) $h(x) = -2\log(x+5)$

(d) $\ell(x) = 0.5x^2 - 2\sin x - 5$

Figure 1: Four real functions of one variable.

Part 2.B

Based on your answer to Part A, which functions out of the list below are convex?

a) $f(x) + g(x)$

b) $\pi h(x)$

c) $2.5f(x) + h(x)$

d) $\max\{f(x)/3, h(x)/7\}$

e) $-\ell(x)$

Solution. The sum of convex functions is convex. The positive multiple of a convex function is convex. The max of convex functions is convex. It follows that b), c) and d) are convex. e) is the negative of a convex function and is therefore concave by definition.

Problem 3

Which ones of the following mathematical programs is *not* a Linear Program? For those that are not Linear Programs, can they be reformulated in linear form?

$$\left.\begin{array}{lrcl} \min & 0.5x_1 + 3x_2 & & \\ & x_1 + 1.35x_2 & = & 2.7 \\ & x_1 & \geq & 0 \\ & x_2 & & \text{free}, \end{array}\right\} \quad \text{(a)}$$

$$\left.\begin{array}{lrcl} \max & 0.5x_1 + 3x_2 & & \\ & 0.99x_1 + x_2 & \geq & 2.7 \\ & x_1 - x_2 & < & 8 \\ & x_1, x_2 & \geq & 0, \end{array}\right\} \quad \text{(b)}$$

$$\left.\begin{array}{lrcl} \min & 0.5x_1 + 3x_2 & & \\ & (0.99x_1 + x_2)/x_2 & \geq & 2.7 \\ & x_1 - x_2 & \leq & 8 \\ & x_1 & \geq & 0 \\ & x_2 & & \text{free}, \end{array}\right\} \quad \text{(c)}$$

$$\left.\begin{array}{lrcl} \min & 0.5x_1 + 3x_2 & & \\ & (0.99x_1 + x_2)/x_2 & \geq & 2.7 \\ & x_1 - x_2 & \leq & 8 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 1.5, \end{array}\right\} \quad \text{(d)}$$

Solution. a) is a Linear Program. b) is not because it contains a strict inequality. c) and d) are not because they contain decision variables at the denominator. d) can be reformulated in linear form by multiplying through the first constraint by x_2, which is a *positive* variable. The same trick does not work for c) because we a priori we do not know the sign of x_2.

Problem 4

Formulate the following problem in algebraic form.

We have m facilities and n customers. Each customer requires $d_j, j = 1, \ldots, n$ units of product, and each facility can produce at most $p_i, i = 1, \ldots, m$ units. Shipping one unit from facility i to customer j costs c_{ij} dollars. Write a Linear Program to minimize the cost of shipping products from the facilities to the customers, meeting the demand of all customers while not exceeding the production capability of any facility. You can assume that we are allowed to ship fractional quantities of product.

Do not be scared by the fact that we have parameters n, m, d_j, p_i, c_{ij} instead of numbers! You can treat them just as you would treat numbers. Start by defining the decision variables. (Hint: we want to decide how many units should go from each facility to each customer.)

Solution. A natural choice for the decision variables of this assignment problem is to consider the decision variables x_{ij} = number of units shipped from the i-th facility to the j-th customer, $i = 1, \ldots, m$, $j = 1, \ldots, n$. The problem can therefore be formulated as follows:

$$\left.\begin{array}{rrcl} \min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & & \\ \text{Demand: } \forall j = 1, \ldots, n & \sum_{i=1}^{n} x_{ij} & = & d_j \\ \text{MaxProduction: } \forall i = 1, \ldots, m & \sum_{j=1}^{n} x_{ij} & \leq & p_i \\ \forall i, j & x_{ij} & \geq & 0, \end{array}\right\} \quad \text{(P)}$$