Optimization Methods in Management Science

MIT 15.053, Spring 2013

PROBLEM SET 2 (SECOND GROUP OF STUDENTS)

Students with first letter of surnames I–Z

DUE: FEBRUARY 21 , 2013

Problem Set Rules:

1. Each student should hand in an individual problem set.

2. Discussing problem sets with other students is permitted. Copying from another person or solution set is *not* permitted.

3. Late assignments will *not* be accepted. No exceptions.

4. The non-Excel solution should be handed in at the beginning of class on the day the problem set is due. The Excel solutions, if required, should be posted on the website by the beginning of class on the day the problem set is due. Questions that require an Excel submission are marked with EXCEL SUBMISSION . For EXCEL SUBMISSION questions, only the Excel spreadsheet will be graded.

Problem 1

(33 points total) Consider the following linear program:

$$
\left.
\begin{array}{llrcr}
\max & x_1 + 2x_2 & & & \\
\text{s.t.:} & & & & \\
\text{Const 1} & x_1 - x_2 & \geq & -2 \\
\text{Const 2} & x_1 + x_2 & \leq & 4 \\
\text{Const 3} & x_1 & \leq & 2.5 \\
\text{Const 4} & x_2 & \leq & 3 \\
& x_1, x_2 & \geq & 0.
\end{array}
\right\}
\qquad \text{(LP)}
$$

(a) (5 points) Graph the feasible region of the LP. Is the feasible region unbounded?

(b) (4 points) Are any of the above constraints redundant? If so, indicate which one(s). (For large linear programs, eliminating redundant constraints can speed up the solution of the linear program.)

(c) (4 points) Solve the LP using the graphical method. Explain your approach.

(d) (4 points) Is there more then one optimal solution? If so give two different solutions. If not, explain using the graphical method why not?

(e) (9 points, 3 points each) Suppose we add the constraint $2x_1 + x_2 \geq \alpha$ to **??**. For which values of α:

   - is the constraint redundant?

- the optimal solution found above is no longer optimal?
- the problem becomes infeasible?

To answer these questions, you should use the graph of the feasible region drawn in Part (a).

(f) (4 points) Replace the objective function $x_1 + 2x_2$ with the objective function $x_1 + \beta x_2$, and compute the values of β for which the point $(2.5, 1.5)$ is optimal.

Problem 2 [1]

(38 points total) A company makes three lines of tires. Its four-ply biased tires produce \$6 in profit per tire; its fiberglass belted line \$4 a tire; and its radials \$8 a tire. Each type of tire passes through three manufacturing stages as a part of the entire production process.

Each of the three process centers has the following hours of available production time per day:

	Process	Hours
1	Modeling	12
2	Curing	14
3	Assembly	16

The time required in each process to produce one hundred tires of each line is as follows:

Tire	Modeling	Curing	Assembly
	Hours per 100 units		
Four-ply	2	3	2
Fiberglass	2	2	1
Radial	4	2	2

(a) (5 points) Write a linear program to determine the optimum product mix for each day's production, assuming all tires are sold.

(b) (10 points) <u>EXCEL SUBMISSION</u> Solve the problem using the simplex algorithm, employing the Excel spreadsheet given with this problem set. Make sure you understand the formulation and the meaning of the variables, and fill in the missing coefficients of the tableau. The spreadsheet that you submit should contain the sequence of tableaus that leads to the optimal tableau *only*. You will then use the same spreadsheet to help through the rest of Problem 2, but make sure you submit a spreadsheet containing the sequence of tableaus leading to the optimal tableau for the *original* data!

(c) (8 points) Using the Excel spreadsheet to carry out the calculations, answer the following question:

(1) What is the initial feasible solution? Give the value for all the decision variables and all the slack variables.

(2) What is the optimal solution? Give its objective function value and the value of all the decision variables.

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[1]This problem is based on Problem 6 of Applied Mathematical Programming, Chapter 2.

(d) (15 points, 5 points each) Using the Excel spreadsheet from the previous point to recompute the optimal solution of the LP, answer the following questions:

  (i) Suppose we increase the number of modeling hours per day from 12 to 13. How much does the profit increase? (Comment: This is the *shadow price* of the constraint on the modeling hours. Read the tutorial on "Sensitivity Analysis in 2 Dimensions" to learn more about shadow prices.)

  (ii) What would be the increase in the profit if we increase the number of assembly hours per day from 16 to 17 (Assume that the number of modeling hours per day is 12.)

  (iii) (Extra credit) Consider the answer to the previous two questions. What relationship do you observe between the shadow price of a constraint and the value of the corresponding slack variable in the optimal solution?

Problem 3 (Second group of students) [2]

(34 points total) A corporation that produces gasoline and oil specialty additives purchases four grades of petroleum distillates, A, B, C, and D. The company then combines the four distillates to make three mixtures (Deluxe, Standard, and Economy) according to specifications of the maximum and/or minimum percentages of grades A, C, or D in each blend, given in Table **??**.

Mixture	Max % allowed for Additive A	Min % allowed for Additive C	Max % allowed for Additive D	Selling price $/gallon
Deluxe	60%	20%	10%	7.9
Standard	15%	60%	25%	6.9
Economy	–	50%	45%	5.0

Table 1: Specifications of the three mixtures.

Supplies of the three basic additives and their costs are given in Table **??**.

Distillate	Max quantity available per day (gals)	Cost $/gallon
A	4000	0.60
B	5000	0.52
C	3500	0.48
D	5500	0.35

Table 2: Supplies and costs of petroleum grades.

We write a linear program to determine the production policy that maximizes profits. To do that, we define twelve decision variables to indicate the gallons of each of the basic petroleum grades (additives) that contribute to the production of the three mixtures. We use the following labels: x_{AD} indicates the quantity of additive A that is used for production of mixture Deluxe, x_{AS} indicates the quantity of additive A that is used for production of mixture Standard, x_{AE} indicates the quantity of additive A that is used for production of mixture Economy. Similarly,

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[2]This problem is based on Problem 11 of Applied Mathematical Programming, Chapter 1.

we use x_{BD}, x_{BS}, x_{BE} to label the quantity of additive B used in each of the three mixtures, x_{CD}, x_{CS}, x_{CE} for additive C, and and x_{DD}, x_{DS}, x_{DE} for additive D. The gallons of mixture Deluxe produced can be computed as $x_{AD} + x_{BD} + x_{CD} + x_{DD}$; similarly for mixture Standard and Economy. The objective function is the total profit, computed as revenues minus costs.

The problem can be formulated as

$$
\begin{array}{rl}
\max & 7.3x_{AD} + 7.38x_{BD} + 7.42x_{CD} + 7.55x_{DD} \\
& 6.3x_{AS} + 6.38x_{BS} + 6.42x_{CS} + 6.55x_{DS} \\
& 4.4x_{AE} + 4.48x_{BE} + 4.52x_{CE} + 4.65x_{DE}
\end{array}
$$

s.t.:

$$
\begin{array}{lrcl}
\text{Availability A}: & x_{AD} + x_{AS} + x_{AE} & \leq & 4000 \\
\text{Availability B}: & x_{BD} + x_{BS} + x_{BE} & \leq & 5000 \\
\text{Availability C}: & x_{CD} + x_{CS} + x_{CE} & \leq & 3500 \\
\text{Availability D}: & x_{DD} + x_{DS} + x_{DE} & \leq & 5500 \\
\text{Mixture D Add A}: & 0.4x_{AD} - 0.6x_{BD} - 0.6x_{CD} - 0.6x_{DD} & \leq & 0 \\
\text{Mixture D Add C}: & -0.2x_{AD} - 0.2x_{BD} + 0.8x_{CD} - 0.2x_{DD} & \geq & 0 \\
\text{Mixture D Add D}: & -0.1x_{AD} - 0.1x_{BD} - 0.1x_{CD} + 0.9x_{DD} & \leq & 0 \\
\text{Mixture S Add A}: & 0.85x_{AS} - 0.15x_{BS} - 0.15x_{CS} - 0.15x_{DS} & \leq & 0 \\
\text{Mixture S Add C}: & -0.6x_{AS} - 0.6x_{BS} + 0.4x_{CS} - 0.6x_{DS} & \geq & 0 \\
\text{Mixture S Add D}: & -0.25x_{AS} - 0.25x_{BS} - 0.25x_{CS} + 0.75x_{DS} & \leq & 0 \\
\text{Mixture E Add C}: & -0.5x_{AE} - 0.5x_{BE} + 0.5x_{CE} - 0.5x_{DE} & \geq & 0 \\
\text{Mixture E Add D}: & -0.45x_{AE} - 0.45x_{BE} - 0.45x_{CE} + 0.55x_{DE} & \leq & 0 \\
& x_{AD}, x_{AS}, x_{AE}, x_{BD}, x_{BS}, x_{BE}, x_{CD}, x_{CS}, x_{CE}, x_{DD}, x_{DS}, x_{DE} & \geq & 0.
\end{array}
$$

You are given an Excel spreadsheet that solves this problem. Suppose that the selling prices are changed to $7.7 per gallon for Deluxe, $6.8 per gallon for Standard, and $4.9 per gallon for Economy. In addition, the marketing department estimates that the demand for Deluxe does not exceed 12000 gallons, therefore the corporation does not want to produce more than this quantity for Deluxe. Assume that all other data remains unchanged. Update the Excel spreadsheet and then answer the following questions:

(a) (10 points) What is the optimal profit and the optimal production-mix?

(b) (5 points) No mixture Economy is produced in the optimal solution. What would the minimum selling price need to be in order for Economy to be worth producing? (Be accurate to within 5 cents).

(c) (6 points) Suppose that you can increase the selling price of Standard. What would be the increase in the profit if the selling price of mixture Standard per gallon increases to $6.8 + s$ for $s = \$ 0.05$, $\$0.1$, and $\$0.15$. (Assume that the selling price of mixtures Deluxe and Economy remain $7.7 and $4.9 per gallon, respectively.) The increase is the difference between the new profit and the profit from Part (a).

(d) (8 points) Based on your answer in Part (c), estimate the profit if the selling price of mixture Standard increases to 7.2. What is the formula for the optimum profit if the selling price of Standard per gallon increased to $\$6.8 + s$? (You may assume that s is between 0.05 and 0.25).

(e) (5 points) Based on your formula in part (d), estimate the contribution (the increase in optimal profit) if the selling price of Standard per gallon increases to 7.5. Use Excel solver to see if your estimation is correct.

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