Massachusetts Institute of Technology
Department of Physics
8.962 Spring 2006

Problem Set 4
Post date: Thursday, March 9th
Due date: Thursday, March 16th

1. Connection in Rindler spacetime

   The spacetime for an accelerated observer that we derived on Pset 2,

$$ds^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \tag{1}$$

   is known as "Rindler spacetime". Compute all non-zero Christoffel symbols for this spacetime. (Carroll problem 3.3 will help you quite a bit here.)

2. Relativistic Euler equation

   (a) Starting from the stress-energy tensor for a perfect fluid, $\mathbf{T} = \rho \vec{U} \otimes \vec{U} + P\mathbf{h}$, where $\mathbf{h} = \mathbf{g}^{-1} + \vec{U} \otimes \vec{U}$, using local energy momentum conservation, $\nabla \cdot \mathbf{T} = 0$, derive the relativistic Euler equation,

$$(\rho + P)\nabla_{\vec{U}} \vec{U} = -\mathbf{h} \cdot \nabla P \ . \tag{2}$$

   (Note: Because both $\mathbf{T}$ and $\mathbf{h}$ are symmetric tensors, there is no ambiguity in the dot products that appear in this problem.)

   (b) For a nonrelativistic fluid ($\rho \gg P$, $v^t \gg v^i$) and a cartesian basis, show that this equation reduces to the Euler equation,

$$\frac{\partial v_i}{\partial t} + v_k \partial_k v_i = -\frac{1}{\rho}\partial_i P \ . \tag{3}$$

   (i, k are spatial indices running from 1 to 3.) What extra terms are present if the connection is non-zero (e.g., spherical coordinates)?

   (c) Apply the relativistic Euler equation to Rindler spacetime for hydrostatic equilibrium. Hydrostatic equilibrium means that the fluid is at rest in the $\bar{x}$ coordinates, i.e. $U^{\bar{x}} = 0$. Suppose that the equation of state (relation between pressure and density) is $P = w\rho$ where w is a positive constant. Find the general solution $\rho(\bar{x})$ with $\rho(0) = \rho_0$.

   (d) Suppose now instead that $w = w_0/(1 + g\bar{x})$ where w_0 is a constant. Show that the solution is $\rho(\bar{x}) = \rho_0 \exp(-\bar{x}/L$. Find L, the density scale height, in terms of g and w_0. Convert to "normal" units by inserting appropriate factors of c — L should be a length.

   (e) Compare your solution to the density profile of a nonrelativistic, plane-parallel, isothermal atmosphere (for which $P = \rho kT/\mu$, where T is temperature and μ is the mean molecular weight) in a constant gravitational field. [Use the nonrelativistic Euler equation with gravity: add a term $-\partial_i \Phi = g_i$, where Φ is Newtonian gravitational potential and g_i is Newtonian gravitational acceleration, to the right hand side of Eq. (3).] Why does hydrostatic equilibrium in Rindler spacetime — where there is no gravity — give such similar results to hydrostatic equilibrium in a gravitational field?

3. Spherical hydrostatic equilibrium

   As we shall derive later in the course, the line element for a spherically symmetric static spacetime may be written

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left[1 - \frac{2GM(r)}{r}\right]^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ ,$$

   where $\Phi(r)$ and $M(r)$ are some given functions. In hydrostatic equilibrium, $U^i = 0$ for $i \in [r, \theta, \phi]$. Using the relativistic Euler equation, show that in hydrostatic equilibrium $p = p(r)$ with

$$\frac{\partial p}{\partial r} = -(\rho + P)\frac{\partial \Phi}{\partial r} \ .$$

4. Converting from non-affine to affine parameterization

   Suppose $v^\alpha = dx^\alpha/d\lambda^*$ obeys the geodesic equation in the form

$$\frac{Dv^\alpha}{d\lambda^*} = \kappa(\lambda^*)v^\alpha \ .$$

   Clearly λ^* is not an affine parameter.

   Show that $u^\alpha = dx^\alpha/d\lambda$ obeys the geodesic equation in the form

$$\frac{Du^\alpha}{d\lambda} = 0$$

   provided that

$$\frac{d\lambda}{d\lambda^*} = \exp\left[\int \kappa(\lambda^*)\, d\lambda^*\right] \ .$$

5. Conserved quantities with charge

   A particle with electric charge e moves with 4-velocity u^α in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_μ. The equation describing this particle's motion can be written

$$u^\beta \nabla_\beta u_\alpha = eF_{\alpha\beta}u^\beta \ ,$$

   where

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \ .$$

   The spacetime admits a Killing vector field ξ^α such that

$$\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0 \ ,$$
$$\mathcal{L}_{\vec{\xi}}\, A_\alpha = 0 \ .$$

   Show that the quantity $(u_\alpha + eA_\alpha)\xi^\alpha$ is constant along the worldline of the particle.