8.871 Lecture Notes 1

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1 Course Outline

This course is an introduction to branes in string theory and their world volume dynamics. The theme of the course will be different that the traditional approach of teaching string theory. Instead of looking at the theory from the point of view of the world-sheet observer, we will approach the problem from the point of view of an observer which lives on a brane. Instead of writing down conformal field theory on the world-sheet and studying the properties of these theories, we will look at various branes in string theory and ask how the physics on their world-volume looks like. This will give a totally different approach than the usual CFT approach on the world-sheet and will give new intuition and new insights on how we should think and understand string theory in various dimensions and supersymmetries. This approach is relatively new. It is a result of the change in thinking the world of string theory had gone through in the past 10 years. During this period researchers in the field begun to understand the importance of branes in string theory and the crucial role they play in various phenomena. The realization that branes are crucial in string theory then led to an opening of a whole new world of research that has to do with the dynamics of supersymmetric gauge theories and quantum field theories on the world volume of branes in various dimensions and supersymmetries.

By the end of this course, we should be able to take an arbitrary configuration of branes and construct a supersymmetric field theory that resembles the Standard Model. We will cover D-branes, which are supersymmetric string solitonic[1] solutions. A Dp-brane is a membrane with p spatial dimensions (or $p+1$ spacetime dimensions). We will study Dp-branes, NS-branes, M-branes, small instantons, and their worldvolume theories and interactions. We'd like to build up an understanding of quantum field theory and supersymmetric gauge theory in the worldvolume of the brane, so that we can eventually make a connection to phenomenology.

2 Charged Strings and Branes

A Dp-brane is a subspace of spacetime in which open strings can end. The D stands for Dirichlet, because the strings which end on Dp-branes have Dirichlet boundary conditions, i.e. the spacetime coordinates of the string endpoints are constrained to lie in the brane. After learning about the open string, one could imagine a theory of open strings with Neumann boundary conditions on all coordinates and no need for auxiliary objects such as Dp-branes. In the perturbative formalism, the first hint that Dp-branes might be essential to string theory appears when studying T-duality, the duality between a string theory compactified on a circle of radius R, and a theory compactified on a circle of radius α'/R. When this duality is applied to open strings, the strings with Neumann conditions get Dirichlet boundary conditions; these strings must necessarily end on Dp-branes.[2] Another way to justify the existence of D-branes is to note that Type IIA and IIB theories contain massless $(p+1)$-forms called Ramond-Ramond (R-R) fields. Then we can posit the existence of objects with $p+1$ spacetime dimensions that are charged under the R-R fields.

[1] We should be careful about using the word solitonic here. Dp-branes have an energy density that scales as the inverse coupling, although solitons usually have energy density that scales as the inverse square coupling.

[2] See, for example, Clifford Johnson's D-Brane Primer (`hep-th/0007170`) p. 61

Now consider the simplest configuration involving a string and a brane. We have an F1 (fundamental) string ending on a Dp-brane. (We use the name F1 to distinguish the original string from a similar one-dimensional object, the D1-brane.) Let's introduce a gauge field $B_{\mu\nu}$ living in the bulk of spacetime. This gauge field is an antisymmetric 2-form, so $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$, and for shorthand we call it $B^{(2)}$. Where does this field come from? Recall that in the spectrum of bosonic closed string states, the massless state $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$ decomposes into a symmetric traceless 2-tensor, a 2-form, and a scalar in SO(D-2). This 2-form reappears in the massless states of any superstring theory. Just as the symmetric traceless 2-tensor corresponds to the graviton and can be used to build up a coherent background (the curvature of spacetime), the 2-form can also provide a background field, $B^{(2)}$. The F1 string typically is charged with respect to $B^{(2)}$. How do we write this in the form of an equation? Let's start with Gauss' Law in 3+1 dimensions:

$$Q = \int_{S^2} \vec{E} \cdot d\vec{a} \ . \tag{1}$$

In a more geometrical form, two of Maxwell's laws (including the differential form of (1)) can be written in terms of the field strength $F^{(2)} = dA^{(1)}$ as follows:

$$d * F^{(2)} = Q\delta^{(3)} \ . \tag{2}$$

Now for the string, we can construct the field strength $H^{(3)} = dB^{(2)}$ and write an equation similar to (2):

$$d * H^{(3)} = Q\delta^{(8)}. \tag{3}$$

Specifically, we can consider an F1-string in Type IIB theory, so that we are living in 10 total spacetime dimensions. So the Hodge dual is taken in 10 dimensions, making $*H$ a 7-form (because $7 = 10 - 3$) and therefore the LHS of (3) is an 8-form. The RHS is also an 8-form because we can write $\delta^{(8)} = \delta(x^1) \wedge \delta(x^2) \wedge \cdots \wedge \delta(x^8)$. By $\delta(x^1)$ we mean the 1-form version; in terms of the basis vector it would be written $\delta(x^1)\ dx^1$.

Now consider the case of the string stretched along the positive x^1 direction and ending on a Dp-brane located at $x^1 = 0$. Our first attempt to write an equation analogous to (3) gives us:

$$d * H^{(3)} = \ Q\delta^{(8)}\theta(x^1) \qquad \text{[naïve guess]} \tag{4}$$

But this equation becomes inconsistent if we take a derivative on both sides: $d(d{*}H^{(3)}) = 0$ but $d(Q\delta^{(8)}\theta(x^1)) = Q\delta^{(9)} \neq 0$. In the case of a D5-brane, we can correct this equation in the following way:

$$d * H^{(3)} = Q\delta^{(8)}\theta(x^1) - \delta^{(4)} *_6 F^{(2)}. \tag{5}$$

The newly subtracted term involves the Hodge dual of a gauge field ($*_6 F^{(2)}$) localized on the brane world-volume ($\delta^{(4)}$). We need

$$d *_6 F^{(2)} = Q\delta^{(5)} \tag{6}$$

so that after taking the derivative of the whole equation, it stays consistent. From this equation we know that $F^{(2)}$ must be a 2-form. The interpretation is that the string is charged under the $B^{(2)}$ field, as in equation (3); furthermore, comparing (6) and (2), the end of the string acts as an "electric" source for the field strength $F^{(2)}$.

Exercise: Generalize the source equation (5) for an F1 string ending on a Dp-brane, for $p = -1...9$.

We can derive (5) as the equation of motion from the action

$$\begin{aligned} S &= \int d^{10}x \left(\frac{1}{2} H^{(3)} \wedge *H^{(3)} + QB^{(2)}\delta^{(8)} - (B^{(2)} \wedge *_6 F^{(2)})\ \delta^{(4)} \right) && (7) \\ &= \int_{\text{spacetime}} \frac{1}{2} H^{(3)} \wedge *H^{(3)} + Q\int_{\text{worldsheet}} B^{(2)} - \int_{\text{worldvolume}} B^{(2)} \wedge *_6 F^{(2)} && (8) \end{aligned}$$

Note that you can go from (5) to (7) by wedging all terms with $B^{(2)}$, integrating over all spacetime, and integrating the first term by parts. The first term in (8) is the kinetic action for the $B^{(2)}$ field that lives in the bulk, so it is an integral over all spacetime. The second term is only an integral over the worldsheet of the string because it represents the charge of the string electrically coupled to $B^{(2)}$. The third term is an integral over the brane worldvolume because it is an interaction between $B^{(2)}$ and $F^{(2)}$ which only takes place on the brane. This last term is necessary in order for the string to end on the brane. As we saw in the discussion above, without this term the string can not end on the brane.

3 Gauge Invariant Action

We would like to preserve gauge invariance under $B^{(2)} \to B^{(2)} + d\Lambda^{(1)}$. Under this gauge transformation, the physics is invariant because $d(d\Lambda^{(1)}) = 0$ and $H^{(3)} \to H^{(3)}$. So any gauge-invariant term in the action will be a function of $H^{(3)}$, not $B^{(2)}$. But what about a term like the integral of B? Let's look at the integral of the pullback of $B^{(2)}$ on some two-dimensional manifold Σ. (Mathematical note: A form is a map from the tangent space of a manifold to $\mathbb{R}$. We define the pullback of a form in Σ by the map from Σ to $\mathbb{R}$ induced by the inclusion map from spacetime to Σ, assuming Σ is a submanifold of spacetime.) The integral looks like

$$I(B) = \int_\Sigma d^2z \; (\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}), \tag{9}$$

where we have changed to brane coordinates, using the ϵ tensor to contract indices because we are dealing with antisymmetric forms. The d^2z indicates integration over worldsheet coordinates. Now let's perform the gauge transformation. We get

$$I(B) \to I(B + d\Lambda) = I(B) + \int_{\partial\Sigma} d\tau \; \Lambda_\mu \frac{dX^\mu}{d\tau} \tag{10}$$

after using Stokes' law to go from an integral of $d\Lambda^{(1)}$ over Σ to an integral of $\Lambda^{(1)}$ over the boundary. We see that in the absence of a boundary, I_B is gauge invariant. In the presence of a boundary (e.g. for the worldsheet of an open string), the gauge invariance is lost. To solve this problem we notice that the end of the string couples electrically to a gauge field $A^{(1)}$ as in equation (6). We will have a term in the action involving

$$\int_{\partial\Sigma} A = \int_{\partial\Sigma} A_\mu \frac{dX^\mu}{d\tau} d\tau \; . \tag{11}$$

Therefore, if we also gauge transform $A^{(1)} \to A^{(1)} + \Lambda^{(1)}$, equation (11) transforms as follows:

$$\int_{\partial\Sigma} A \to \int_{\partial\Sigma} A + \int_{\partial\Sigma} d\tau \; \Lambda_\mu \frac{dX^\mu}{d\tau}. \tag{12}$$

As a result, the combined action

$$\int_\Sigma B - \int_{\partial\Sigma} A \tag{13}$$

is gauge invariant. This means that if we started with the combination $B^{(2)} - F^{(2)}$ in (9), we would get

$$\begin{aligned} I(B-F) = I(B) - I(F) &= \int_\Sigma B - \int_\Sigma F = \int_\Sigma B - \int_\Sigma dA \qquad (14) \\ &\to I(B) + \int_{\partial\Sigma} d\tau \; \Lambda_\mu \frac{dX^\mu}{d\tau} - \int_{\partial\Sigma} A - \int_{\partial\Sigma} d\tau \; \Lambda_\mu \frac{dX^\mu}{d\tau} \\ &\to I(B-F) \qquad (15) \end{aligned}$$

Thus we learn that any gauge-invariant action is a function of $B^{(2)} - F^{(2)}$, not $B^{(2)}$.

4 Massless fields in 11d SUGRA

In this course we will start with theories with the highest supersymmetry (32 supercharges[3]) and then successively divide by two. The following are theories with 32 supercharges: 11 dimensional supergravity (SUGRA), Type IIA and Type IIB string theory in flat 10 dimensions. We will first review the massless fields in 11d SUGRA. They form representations of the little group for massless states, SO(9). First let's state the number of degrees of freedom in irreps of the massless little group in an arbitrary dimension d.

$$\text{Symmetric, traceless 2-tensor (graviton)} \quad \rightarrow \quad \frac{(d-1)(d-2)}{2} - 1 = \frac{d(d-3)}{2} \tag{16}$$

$$\text{p-form} \quad \rightarrow \quad \binom{d-2}{p} \tag{17}$$

$$\text{Spinor} \quad \rightarrow \quad 2^{\left[\frac{d-3}{2}\right]} \tag{18}$$

$$\text{Gravitino} \quad \rightarrow \quad (d-3)\, 2^{\left[\frac{d-3}{2}\right]} \tag{19}$$

Now we can evaluate the above expressions at $d = 11$ for the massless supergravity multiplet. First we have $g_{\mu\nu}$, the graviton, with $\frac{9\cdot 10}{2} - 1 = 44$ bosonic degrees of freedom. Next we have a 3-form $C^{(3)}$ which has $\binom{9}{3} = 84$ components. Finally, the gravitino ψ_μ has $8 \cdot 2^4 = 128$ components. This means that the number of bosonic degrees of freedom (44+84) equals the fermionic degrees of freedom (128), as it should in a supersymmetric theory.

[3]The same as the old name $\mathcal{N} = 8$, which applies to four dimensions only. The modern label "32 supercharges" does not depend on the number of dimensions.