8.821 String Theory
Fall 2008

8.821 F2008 Lecture 21: Confinement, continued

Lecturer: McGreevy

January 30, 2009

- Confining geometry and mass gap (main lesson: mass gap $\longrightarrow$ smooth minimum of the wrap factor in the dual geometry)

- Black hole mechanics

Last time we were talking about attempting to find a gravity dual of a deformation to 3D YM theory (which we get from the $\mathcal{N} = 4$ on a thermal circle[1]). In these compactifications, the 3D YM coupling g_3 is of order of the mass of the scalars

$$g_3 \approx m_X \approx \frac{g_4 N}{R_y} \tag{1}$$

where $g_4 N$ is the 4D t' Hooft coupling. The 3D theory is not just YM but it also has a bunch of scalars and this obscures our intuition about the theory.

We can do a similar thing to get 4D YM theory by putting $ND4$ branes (at low energies) on a thermal circle with a radius R_y and again Fermions will have APBCs. The $ND4$ theory is a gauge theory in 5D with 16 supercharges and its well defined at low energies. The boundary conditions on the thermal circle will break supersymmetry and the fermions will get 4D masses of order $1/R_y$ while bosons will get masses at one loop of order $m_X^2 \approx g_4^2 N m_\psi^2 \approx \lambda_4/R_y^2$. At energies $E \approx 1/R_y$, the 4D and 5D YM couplings are related by $1/g_4^2 = R_y/g_5^2$ which implies that $m_X \approx \sqrt{\lambda_4}/R_y$. The relation between the the 4D and 5D YM couplings comes from integrating over the extra dimension in the compactified theory.

The nice thing about the above construction is that if at the UV scale $(E = 1/R_y)$, the t' Hooft coupling $\lambda_4 \ll 1$, then the theory is asymptotically free and we can calculate the β function perturbatively. There will be a dynamically genrated mass scale Λ such that

$$\Lambda_{\text{QCD}} \approx \frac{e^{-1/\lambda_4}}{R_y} \ll \frac{1}{R_y} \ll m_X \approx \frac{\sqrt{\lambda_4}}{R_y} \tag{2}$$

Notice that $1/R_y$ plays the role of a UV. Its like putting the 4D YM theory on a lattice where R_y plays the role of a lattice spacing. If we probe the theory with energies of order $1/R_y$ we will see a 5D theory (the UV completion).

---

[1] Remember that thermal circle means periodic boundary conditions for Bosons and antiperiodic boundary conditions for Fermions

Notice also that when $E < 1/R_y$, we will have a pure YM theory. The previous analysis was for $\lambda_4 \ll 1$, the gravity dual has a curvature which goes like $1/\lambda_4$ and we can't trust the supergravity description. The gravity dual is valid when $\Lambda_{\text{QCD}} \gg \text{UV}$, i.e, when λ_4 is large. At large λ_4 the QFT isn't weakly coupled at the cutoff, so there is never a regime where the usual perturbative calculation of asymptotic freedom is valid.

Back to the 3D case

Despite that the 4D YM theory is a little bit cleaner in the weak coupling limit than the 3D case but the details of the gravity dual in the 3D is simpler. To find a gravity dual we would like some solution which satisfies Einstein equations in the bulk with a negative cosmological constant

$$R_{ab} - \frac{1}{2}g_{ab} = \Lambda g_{ab} \tag{3}$$

which asymptotes at the boundary to $R^2 \times S_y^1$. There are two solutions which satisfy these requirements
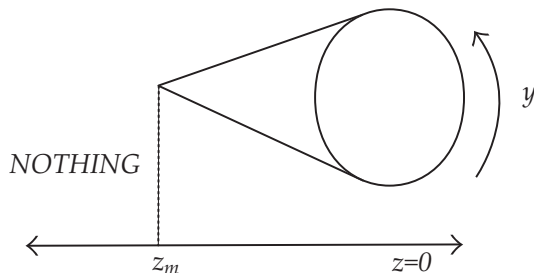
(a) Compactify a circle on AdS

(b) The other solution which is slightly more interesting is

$$ds^2 = \frac{L^2}{z^2}\Big(- dt^2 + d\vec{x}^2 + f dy^2 + \frac{dz^2}{f} \Big), \tag{4}$$

where

$$f = 1 - \frac{z^4}{z_m^4}, \qquad y \cong y + 2\pi R_y \tag{5}$$

Note that f has a zero at z_m where the radius of the y circle shrinks to zero. For a random value of z there will a canonical deficit angle. For $z < z_m$, there is NOTHING, i.e, when the circle shrinks the space ends. We can make the solution much more well defined if we demand that there is no singularity at $z = z_m$. Locally near $z = z_m$,

$$ds^2 = \kappa \rho^2(z)dy^2 + d\rho^2 + \cdots , \tag{6}$$

where,

$$\rho = \sqrt{z_m(z - z_m)}, \qquad \kappa = \frac{2}{z_m} \quad \text{"Surface gravity"} \tag{7}$$

2

This form of the metric makes it clear that the shrinking of the y circle is like a cone where the y direction is fibred over the ρ direction which shrinks to zero at $z = z_m$. To avoid the conical deficit, we chose the periodicity of y such that

$$y \cong y + 2\pi\kappa^{-1} \longrightarrow \theta \equiv \frac{y}{\kappa} \cong \theta + 2\pi \tag{8}$$

and the metric becomes

$$ds^2 = \rho^2 d\theta^2 + d\rho^2 + \cdots \tag{9}$$

and $\rho = 0$ is the origin of the polar coordinates. If we chose a different periodicity there will be a conical deficit angle.

To match the boundary conditions and since we know that y is periodic with a period $2\pi\kappa^{-1}$ where $\kappa = 2/z_m$, then $z_m = 2R_y$. The topology of the boundary is $\mathbb{R}^{2+1} \times S^1$ and the topology of the bulk is $\mathbb{R}^{2+1} \times D_2(z, y)$, where $D_2(z, y)$ is a disk with angular coordinate y and the y circle is contractible in the bulk which implies APBCs on the bulk Fermions.

Side Comments

• The importance of APBCs on the circle for the existence of a solution where the geometry ends is quit reminiscent of the instability of the Kaluza-Klein vacuum [1] to the appearance of a bubble of nothing, which can only exist if we put APBCs on the Fermions, and is similar to the region $z < z_m$ here.

• We should think of the two solutions which we described as two saddle point contributions to the path integral of gravity with the specified boundary conditions. The solution which dominates is the one which has a smaller action when evaluated on the saddle point. John claims that solution (b), which has a mass gap, dominates over (a) and its free energy wins for all values of R_y.
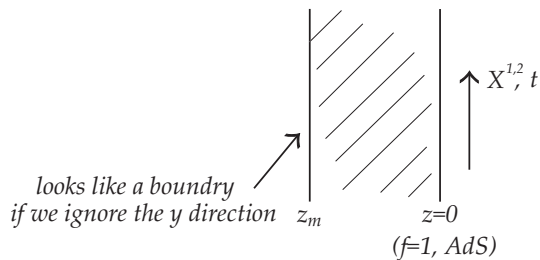
• The warp factor in (b) has a minimum, $W_{\min}(z) \approx 1/z_m^2$,

$$ds^2 = W(z)^2 dx^\mu dx_\mu + \frac{dz^2}{z^2}. \tag{10}$$

Heuristically, the reason why there is a mass gap is that if we have a particle propagating in this space it will fall to this minimum value of the warp factor which acts like a gravitational potential, i.e, lowest energy states are localized around $z = z_m$ unlike the Poincaré AdS case where particles fell to the Poincaré horizon. This statement will be made more precise next time.

The spectrum of the 2+1 theory from the 2 point function

Lets think about the two point function in momentum space of some gauge invariant operator such as the glueball operator $\mathcal{O} = \text{Tr}FF$. In a *unitary* field theory, the two point function has the spectral representation

$$G_2(k) = \langle \mathcal{O}(k)\mathcal{O}(-k)\rangle = \int_0^\infty d\mu \rho(\mu)\frac{1}{\omega^2 - \vec{k}^2 - \mu^2}, \tag{11}$$

where $\rho(\mu)$ is the density of states in the field theory which couples to the operator $\mathcal{O}$,

$$\rho(\mu) = \|\langle\mu|\mathcal{O}(-\omega,-k)|0\rangle\|^2 \tag{12}$$

and unitarity implies that $\rho(\mu) \geq 0$. If there is a mass gap then ρ will have support only above some $\mu_{\min}$,

$$\rho(\mu < \mu_{\min}) = 0. \tag{13}$$

If there is a discrete spectrum then

$$\rho(\mu) = \sum_i c_i \delta(\mu - M_i) \tag{14}$$

this need not be the case if there is no mass gap.

Consider a bulk scalar ϕ with mass m in this confining background with the action

$$S = \int \sqrt{g}\big((\partial\phi)^2 + m^2\phi^2\big), \tag{15}$$

where we kept only the quadratic part since we are interested in the two point function. For example, ϕ could be the dilaton with its zero momentum mode on the sphere ($m = 0$) couples to the glueball $\mathcal{O} = \text{Tr}FF$.

To compute G_2 we use the following ansatz for ϕ

$$\phi = \phi_k(z)e^{-i\omega t + i\vec{k}\cdot\vec{x}} \tag{16}$$

and without loss of generality we will set $\vec{k} = 0$. This is justified by the fact that as long as there is no instability then modes with finite $\vec{k}$ will have higher energies than modes with $\vec{k} = 0$. Now we plug the above ansatz for ϕ in the bulk wave equation

$$0 = |g^{00}|\omega^2\phi_k + \frac{1}{\sqrt{g}}\partial_z(\sqrt{g}g^{zz}\partial_z\phi_k)m^2L^2\phi_k \tag{17}$$

In the interior, $z = z_m$ is like the origin of polar coordinates and and regularity at z_m implies that

$$\partial_z\phi|_{z=z_m} = 0 \tag{18}$$

this condition replaces the condition of finiteness at the Poincaré horizon for the usual AdS. At the boundary, we should impose that the solution is normalizable. This because we want to find the spectrum of the theory in the vacuum without any sources. Remember that the general solution of the wave equation near the boundary behaves as

$$\phi \sim z^{\Delta_-}\phi_0 + z^{\Delta_+}A, \tag{19}$$

where Δ_- is the dimension of the source ϕ_0, Δ_+ is the dimension of the operator to which ϕ_0 couples, and we showed before that

$$A \sim \langle \mathcal{O} \rangle_{\phi_0}. \tag{20}$$

If the source is small then we can use linear response (or by Taylor expanding A in ϕ_0) to write

$$A \sim \langle \mathcal{O} \rangle_{\phi_0} + O(\phi_0^2) \sim G_2 \phi_0 \Longrightarrow G_2 \sim \frac{A}{\phi_0}, \tag{21}$$

the above equation says that if we find a normalizable solution with $\phi_0 = 0$ for some value of the frequency and $A \neq 0$, then the Green's function G_2 will have a pole. There is a general argument that poles in G_2 never arise from A [2].

The general lesson from the above discussion is that to find poles of the Green's function, we look for normalizable solutions of the wave equation.

The wave equation after plugging the metric takes the form (we will drop the subscript k on ϕ)

$$\omega^2 \phi = -z^{D-1} \partial_z (z^{D-1} f \partial_z \phi) + \frac{m^2 L^2}{z^2} \phi = \mathcal{L} \phi \tag{22}$$

for some differential operator $\mathcal{L}$. This is an eigenvalue equation which can be written as a Schrödinger equation (Pset 4). Poles in the Green's function G_2 will occur at the special values of $\omega_\star^2 = M_i^2$ for which $\mathcal{L}$ has normalizable eigenfunctions subject to the boundary condition at $z = z_m$. Multiplying both sides of (22) with ϕ and integrating by parts we get

$$\underbrace{\omega_\star^2 \int_0^{z_m} dz \sqrt{g} |g^{00}| \phi^2}_{\geq 0 \text{ and } = 0 \text{ iff } \phi = 0} = \int_0^{z_m} dz \sqrt{g} \left[g^{zz} (\partial_z \phi)^2 + m^2 L^2 \phi^2 \right] > 0 \tag{23}$$

The spectrum is as if the bulk geometry were compact (in that there are discrete energy levels), but there is no zeromode since the zeromode is not normalizable in AdS.

Mass gap: *Without z_m, we would have an oscillating solution near $z = \infty$, i.e. continuous spectrum down to $\omega = 0$.*

Region of validity: If $\lambda = g_{\text{YM}}^2(\mathcal{N} = 4)N \ll 1$, we have a weakly coupled 3D YM at $E \approx 1/R_y$ which is UV completed by $\mathcal{N} = 4$ SYM on S^1. The gravity description in this case is bad. For $\lambda \gg 1$, the gauge theory is strongly coupled and the gravity description is good.

$r \ll (\text{mass gap})^{-1}$

z

$z = z_m$
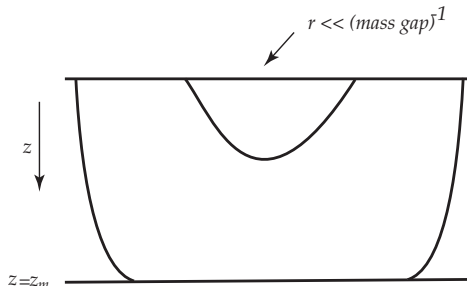
References

[1] E. Witten, "Instability of the Kaluza-Klein Vacuum", Nucl.Phys. B195:481,1982.

[2] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications," JHEP **0209**, 042 (2002) [arXiv:hep-th/0205051].