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8.821 String Theory
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8.821 F2008 Lecture 09: Preview of Strings in $\mathcal{N} = 4$ SYM; Hierarchy of Scaling dimensions; Conformal Symmetry in QFT

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1 Emergence of Strings from Gauge Theory

Continuing from the previous lecture we substantiate the correspondence between the operators in $\mathcal{N} = 4$ SYM theory with the excitations of some string theory. We saw that the primary chiral operators (i.e. primary operators in the conformal field theory which also belong to the short multiplets of the supersymmetry) correspond to the SUGRA modes but we didn't discuss what happens when the operators in the gauge theory are in the long multiplet. Thus consider a more generic operator (residing in a long supersymmetry multiplet) of the matrix QFT in the large N limit

$$O(x) = tr(XXXXYXXY....YX) \tag{1}$$

We consider the limit $N \gg J \gg 1$ where J is the number of entities in the above product. Cyclicity of the trace implies that the structure of the above operator has the symmetry of a closed loop. In fact the above operator corresponds to creation operator for excited string states with X's and Y's the fields living on the worldsheet. Further, in the large J limit, J corresponds to the angular momentum of the string excitations and in fact this relation could be used to reconstruct the string theory.

Having sketched the correspondence between the various operators of the SYM guage theory with SUGRA modes or string excitations, let us organize this knowledge to get a better perspective on when it could be useful.

- The mass of the SUGRA mode is given by $m^2_{SUGRA} = 1/L^2_{AdS}$. Using the AdS-CFT correspondence $m^2_{SUGRA}L^2_{AdS} = \Delta(\Delta - 4)$ where Δ is the scaling dimension of the corresponding chiral primary operator in the SYM theory. This implies that $\Delta_{SUGRA} \sim N^0\lambda^0$.
- Moving on to the excited string states $m^2_{string\ states} \sim 1/\alpha' \sim \sqrt{\lambda}/L^2_{AdS}$ where $1/\alpha'$ is string tension. Thus the corresponding scaling dimension equals $\Delta_{string\ states} \sim N^0\lambda^{1/4}$.
- Finally, there are D-branes in the string theory which correspond to baryonic states in the

gauge theory. Their mass is given by [1]

$$m^2_{D-brane} \sim 1/\alpha' g_s^2 \sim \sqrt{\lambda}/g_s^2 L^2_{AdS} \Rightarrow \Delta_{D-brane} = N\lambda^{1/4}.$$

Clearly, the scaling dimension Δ for the different states (SUGRA, strings, D-branes) has distinctive dependence on N and λ. *This hierarchy of Δ's is what a QFT needs to have a weakly coupled gravity description without strings.* For example the $\lambda \to \infty$ limit removes all states except SUGRA modes.[2]

There are various forms of AdS-CFT conjecture. For example, one may believe that it only holds in the limit $\lambda \to \infty, N \to \infty$ (yielding classical gravity) or perhaps also at finite λ and $N \to \infty$ limit (when one obtains classical strings on small AdS radius). Remarkably, all evidence till date points to a much stronger statement that it holds for all N and all λ.

2 Conformal Symmetry in QFT

This is a worthy subject in itself and study of conformal invariance is relevant for understanding both UV and IR limit of various QFT's and also for the worldsheet theory of the strings in the conformal gauge. As you may already know (and we recapitulate it below) that conformal symmetry in two dimensions is very special due to existence of infinite number of conserved currents. Therefore it pays to understand which aspects of a conformal field theory (CFT) are particular to $D = 2$ and which apply to any dimension $D > 2$. We have already seen some of the constraints due to the requirements of Lorentz invariance and SUSY on the QFT's and expectedly, requirement of conformal invariance makes it even more constrained. This is also the last stop in our ruthless program to evade the loopholes in the Coleman-Mandula theorem.

Some of the useful references are the book *Conformal Field theory* by Di Francesco, Mathieu and Senechal and the articles by Callan, Ginsparg and portions of MAGOO (links posted on the course webpage).

2.1 The Conformal Group

We define CFT by a list of operators and their Green functions which satisfy certain constraints described below. Succintly, a CFT is a theory with symmetries generated by conformal group . Obviously, it behooves us to define conformal group which we do now. Please note that the route we follow is not a good one for non-relativistic CFT's.

[1] The power of λ here depends on the specific geometry of the D-brane in question, *i.e.* which of the dimensions of the bulk the D-brane is wrapping.

[2] It is worth noting here that gauge theories in general also contain non-local operators such as Wilson loops:

$$\mathrm{tr}\ Pe^{i\int_C A}$$

where C is some closed curve in the space on which the field theory lives. It can be thought of as the phase acquired by a charged particle dragged along the specified path by an arbitrarily powerful external force. The connection between these operators and strings turns out to be quite direct, as one might expect from the relationship between charges on D-branes and the ends of open strings.

Recall that isometry group of a spacetime with coordinates x^μ and metric $g_{\mu\nu}$ is the set of coordinate transformations which leave the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ unchanged. Conformal group corresponds to a bigger set of transformations which preserve the metric up to an overall (possibly position-dependent) rescaling: $ds^2 \rightarrow \Omega(x)ds^2$. Thus Poincare group (which is the group of isometries of flat spacetime) is a subgroup of the conformal group with $\Omega(x) = 1$. Conformal transformations could also be seen as the successive application of a coordinate transformation $x \rightarrow x'$, $g_{\mu\nu} \rightarrow g'_{\mu\nu}$ which preserve $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ followed by a Weyl rescaling which takes $x' \rightarrow x$ so that ds^2 is *not* preserved.

Conformal transformations preserve the angles between vectors, hence the name:

$$cos(\theta) = \frac{v^\mu w_\mu}{\sqrt{v^\mu v_\mu}\sqrt{w^\mu w_\mu}} \rightarrow cos(\theta') \tag{2}$$

Consider the space-time $\mathbb{R}^{p,q}$ with a constant metric $g_{\mu\nu} = \eta_{\mu\nu}$ and let's look at the piece of infinitesimal coordinate transformations connected to the identity:

$$\begin{aligned} x^\mu &\rightarrow x^\mu + \epsilon^\mu(x) && (3)\\ g_{\mu\nu} &\rightarrow \eta_{\mu\nu} - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) && (4) \end{aligned}$$

Note that above there are no terms with derivative(s) of metric since the metric is constant. The requirement of conformal invariance implies that

$$\begin{aligned} (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) &= f(x)\eta_{\mu\nu} && (5)\\ &= \frac{2}{d}\,\partial_\rho\epsilon^\rho\,\eta_{\mu\nu} && (6) \end{aligned}$$

where the equality $f(x) = \frac{2}{d}\,\partial_\rho\epsilon^\rho$ is obtained by taking trace of $\partial_\mu\epsilon_\nu$ on both sides of the eqn. 5. Applying an extra derivative ∂_ρ on 5 gives

$$\Box\epsilon_\nu + (1 - \frac{2}{d})\partial_\nu(\partial_\rho\epsilon^\rho) = 0 \tag{7}$$

Pausing for a moment, we note that $d = 2$ is clearly special since $\Box\epsilon_\nu = 0$ implies that *any* holomorphic (antiholomorphic) function $\epsilon(z)$ $(\epsilon(\bar{z}))$ of complex coordinates $z = x^1 + ix^2$ would correspond to a conformal transformation. A little more thought leads to the result that conformal algebra is *infinite dimensional* in two dimensions. Applying one more partial derivative ∂_μ on this equation and applying $\Box$ on eqn. 5 and using the both equations thus obtained together gives

$$(2 - d)\partial_\mu\partial_\nu\partial_\rho\epsilon^\rho = \Box\partial_\rho\epsilon^\rho\eta_{\mu\nu} \tag{8}$$

Contracting with $\eta_{\mu\nu}$ implies $\Box\partial_\rho\epsilon^\rho = 0 = \partial_\mu\partial_\nu\partial_\rho\epsilon^\rho$ and thus $\epsilon^\mu(x)$ is at most a quadratic function of x.

Let's organize $\epsilon^\mu(x)$ by its degree in x:

- Degree zero:

$$\text{Translation} : \epsilon^\mu(x) = a^\mu \tag{9}$$

- Degree one:

$$\begin{aligned} \text{Rotation} : \epsilon^\mu(x) &= \omega^\mu_\nu x^\nu \ \text{ with } \ \omega_{\mu\nu} = -\omega_{\mu\nu} && (10) \\ \text{Dilatation} : \epsilon^\mu(x) &= \lambda x^\mu && (11) \end{aligned}$$

- Degree two:

$$\text{Special Conformal} : \epsilon^\mu(x) = b^\mu \vec{x}.\vec{x} - x^\mu\,\vec{b}.\vec{x} \tag{12}$$

The corresponding generators for the above transformations can now be immediately written down:

$$\begin{aligned} \text{Translation} : P_\mu &= -i\partial_\mu && (13) \\ \text{Rotation} : M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && (14) \\ \text{Dilatation} : D &= ix^\mu\partial^\mu && (15) \\ \text{Special Conformal} : C_\mu &= -i(\vec{x}.\vec{x}\partial_\mu - 2x_\mu\vec{x}.\vec{\partial}) && (16) \end{aligned}$$

The commutation relations for the above generators are obtained as:

$$\begin{aligned} [P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) && (17) \\ [C_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}C_\nu - \eta_{\rho\nu}C_\mu) && (18) \\ [D, M_{\mu\nu}] &= 0 && (19) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= (\eta_{\nu\rho}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) && (20) \\ [D, P_\mu] &= iP_\mu && (21) \\ [D, C_\mu] &= -iC_\mu && (22) \\ [C_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}) && (23) \end{aligned}$$

The first four equations mean that P^μ and C^μ transform as vectors under Lorentz transformations while D is a scalar and $M_{\mu\nu}$ is a rank-2 tensor. The next two equations mean that P^μ and C^μ act as raising and lowering operator for the eigenvectors of dilation operator D.

Interestingly, recombining conformal generators in suitable way reveals the very important and remarkable result that the conformal group in $\mathbb{R}^{p,q}$ is isomorphic to $SO(p+1,q+1)$!. First we do simple counting to check that the number of generators match correctly. The total number of generators of conformal group in $d(=p+q)$ dimensions are d (translation) $+d(d+1)/2$ (rotation) $+d$ (special conformal) $+1$ (dilatation) which sum up correctly to $(d+1)(d+2)/2$, the number of generators for $SO(p+1,q+1)$. To see the exact correspondence, define the following generators:

$$J_{\mu\,\nu} = M_{\mu\nu} \tag{24}$$

$$J_{\mu\,d} = \frac{1}{2}(P_\mu - C_\mu) \tag{25}$$

$$J_{\mu\,d+1} = \frac{1}{2}(P_\mu + C_\mu) \tag{26}$$

$$J_{d+1\,d} = D \tag{27}$$

where $\mu \in \{0,1,...d-1\}$. These new generators follow the $SO(p+1,q+1)$ commutation relations:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{28}$$

where $a,b,c,d \in \{0,1,...d+1\}$. Since the conformal algebra closes, it is possible to exponentiate the infinitesimal transformations to obtain finite ones:

$$\text{Translation}: x \rightarrow x'^{\mu} = x^\mu + a^\mu, \qquad \Omega(x) = 1 \tag{29}$$

$$\text{Lorentz}: x \rightarrow x'^{\mu} = \Lambda^\mu{}_\nu x^\nu, \qquad \Omega(x) = 1 \tag{30}$$

$$\text{Dilatation}: x \rightarrow x'^{\mu} = \lambda x^\mu, \qquad \Omega(x) = \frac{1}{\lambda^2} \tag{31}$$

$$\text{Special conformal}: x \rightarrow x'^{\mu} = \frac{x^\mu + b^\mu \vec{x}.\vec{x}}{1 + 2\vec{b}.\vec{x} + \vec{b}.\vec{b}\,\vec{x}.\vec{x}}, \qquad \Omega(x) = (1 + 2\vec{b}.\vec{x} + \vec{b}.\vec{b}\,\vec{x}.\vec{x})^2 \tag{32}$$

The last transformation ('special conformal') may not seem very easy to visualize. One simpler way to rewrite its finite form is

$$\frac{x'^{\mu}}{\vec{x'}.\vec{x'}} = \frac{x^\mu}{\vec{x}.\vec{x}} + b^\mu \tag{33}$$

That is, it is nothing but a inversion followed by a translation followed again by inversion. As an aside, an alternative definition of the conformal group is that it is the smallest group which contains both Poincare and inversion operations.

2.2 Representation Theory

In a general QFT one is dealing with a large set of fields, their derivatives and products and it is difficult to organize various fields/operators. However in a CFT i.e. in a conformally invariant QFT there is a systematic way to organize various operators. Namely, we first diagonalize the dilatation operator D and label the various operators by the their eigenvalues Δ when being acted upon by D. Note that D commutes with $M^{\mu\nu}$ and hence we can still specify representations of the Lorentz algebra.

Considering a spinless field for simplicity, under a dilatation

$$\phi(x) \to \phi(x') = \lambda^{\Delta}\phi(0) \tag{34}$$

which implies

$$[D, \phi(0)] = -i\Delta\phi(0) \tag{35}$$

of special interest is a **conformal primary operator** transforms under conformal transformations as:

$$\phi(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} \phi'(x') \tag{36}$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the conformal transformation of the coordinates which is related to the scale factor Ω by

$$\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-d/2} \tag{37}$$

One must note that *not every field transforms this way even if it has definite scaling dimension.*

It turns out that the unitary representations of conformal algebra always have a lowest Δ whose value is determined by the dimension of the spin. Physically this makes sense since correlators $\langle\phi(x)\phi(0)\rangle$ of the fields are not expected to diverge as $|x| \to \infty$. Readers familiar with state-operator correspondence (to be discussed in the following lecture(s)) may note that this could also be seen as a consequence of the fact that the spectrum of any unitary theory is bounded from below.

Since C_μ acts as a lowering operator for the scaling dimension (eqn. 22) , it annihilates the state $\phi_{lowest\,\Delta}$. This operator of lowest Δ can be seen to be conformal primary according to the definition above (the Callan article does this somewhat explicitly). Now the strategy for organizing the whole spectrum of operators might be obvious: we label the various representations of the CFT by $\phi_{lowest\,\Delta}$ and build the complete multiplet by acting with P_μ which acts as raising operator for scaling dimension Δ (eqn. 21) .

As a final remark for this lecture: please note that in general primary operators are not eigenfunctions of the Hamiltonian H since the dilatation operator doesn't commute with H. As a consequence $p^\mu p_\mu$ which is the Casimir of the Poincare group is not Casimir of the full conformal group.