8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2014

Lecture 6

1.3: HOLOGRAPHIC PRINCIPLE

If we do treat black hole as an "ordinary" QM object, an important implication would be the holographic principle.

Consider an isolated system mass E and entropy S_0 in an asymptotic flat spacetime. Let A be the area of the smallest sphere hat encompasses the system, and M_A to be the mass of a black hole with the same horizon area, we must have $E < M_A$, otherwise the system would be already a black hole.

Now add $M_A - E$ energy to the system (keeping A fixed), we shall obtain a black hole with mass M_A, since

$$S_{BH} \geqslant S_0 + S'$$

where S' is the entropy of added energy, we conclude that

$$S_0 \leqslant S_{BH} = \frac{A}{4\hbar G_N}$$

i.e. the maximal entropy inside a region bounded by area A is

$$S_{max} = \frac{A}{4\hbar G_N}$$

Recall the definition of entropy in quantum statistical physics:

$$S = -\operatorname{Tr}\rho\log\rho$$

where ρ is the density matrix for the state of a system. For a system with N-dimensional Hilbert space

$$S_{max} = \log N$$

Hence the "effective" dimension of the Hilbert space for a system inside a region of area A is bounded by

$$\log N \leqslant \frac{A}{4\hbar G_N} = \frac{A}{4l_p^2}$$

Remarks:

1. For a system of n spins, the Hilbert space dimension $N = 2^n$.
2. The dimension of $\mathcal{H}$ for a single harmonic oscillator is infinite. But for a quantum system with finite number of degrees of freedom (d. o. f.), the dimension of $\mathcal{H}$ below some finite energy scale is always finite, that's why we have "effective" in the above description.
3. In typical physical systems,
   $$\text{number of d. o. f.} \sim \log N$$
   thus we can write
   $$\text{number of d. o. f. of any quantum gravity system} \leqslant \frac{A}{4l_p^2}$$
4. The bound is violated in non-gravitational systems whose number of d. o. f. (or $\log N$) is proportional to the volume rather than area of the system. *e.g.* for a lattice of spins with lattice spacing a, total number of spins is $\frac{V}{a^3} = \frac{A}{a^2}\frac{L}{a} \gg \frac{A}{l_p^2}$ for large enough L. Also $N = 2^{V/a^3}$, we have $S_{max} = \frac{V}{a^3}\log 2 \geqslant S_{BH}$ for large enough volume. In other words, quantum gravity leads to a huge reduction of d. o. f.

Holographic principle: In quantum gravity, a regime of boundary area A can be fully described by no more than $\frac{A}{4\hbar G_N} = \frac{A}{4l_p^2}$ degrees of freedom, *i.e.* degree of freedom per Planck area. Black hole brings quantum gravity to a macroscopic level.

1.3: LARGE N EXPANSION OF GAUGE THEORIES

We now look at clues to holographic duality from field theory side.

Consider QCD which can be described as $SU(3)$ gauge theory with fundamental quarks. The Lagrangian reads

$$\mathcal{L} = \frac{1}{g_{YM}^2}\left[-\frac{1}{4}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - i\overline{\Psi}(\not{D} - m)\Psi\right]$$

where $D_\mu = \partial_{mu} - iA_\mu$, A_{mu} are 3×3 Hermitian matrices and can be expressed as $A_\mu = A_\mu^a T^a$, with $T^a \in SU(3)$. In such a theory, coupling becomes strong in IR ($\Lambda_{QCD} \sim 250$ MeV), there is no small parameter to expand. It is still an open problem to derive IR properties of QCD from the first principle.

t' Hooft in 1974 suggested take number of color $N = 3$ as a parameter, *i.e.* promote A_μ to $N \times N$ hermitian matrices and consider $N \to \infty$ limit and do a $\frac{1}{N}$ expansion. It is an ingenious idea, unfortunately, QCD still cannot be solved to leading order in the large N limit. Surprisingly, there is an correspondence between the large N gauge theory and the string theory. The key is the fields are matrices. As an illustration, we will consider a scalar theory:

$$\mathcal{L} = -\frac{1}{g_2}\operatorname{Tr}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{4}\Phi^4\right]$$

where g is the cooling constant, $\Phi(x) = \Phi^a_b(x)$ is a $N\times N$ hermitian matrix, *i.e.* ${\Phi^a}_b{}^* = {\Phi^b}_a$. In terms of components

$$\mathcal{L} = -\frac{1}{g^2}\left[\frac{1}{2}(\partial_\mu {\Phi^a}_b)(\partial^\mu {\Phi^b}_a) + \frac{1}{4}{\Phi^a}_b{\Phi^b}_c{\Phi^c}_d{\Phi^d}_a\right]$$

$\mathcal{L}$ is invariant under $U(N)$ "global" symmetry:

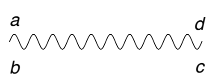
$$\Phi(x) \to U\Phi(x)U^\dagger$$

where U is any constant $U(N)$ matrices.

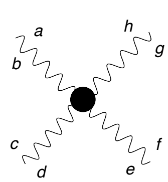
Remarks

1. It is a theory of N^2 scalar fields.
2. One can also consider other types of matrices, *e.g.* $N\times N$ real symmetric matrix, the corresponding symmetry will be $SO(N)$.
3. One could also introduce gauge fields to make the $U(N)$ symmetry local (this point is important and we will discuss it later).

Here we list the Feynman rules for this theory: The propagator:

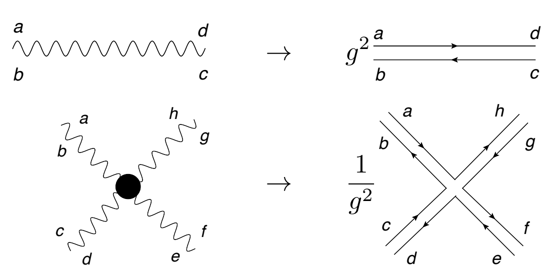
$\langle {\Phi^a}_b(x){\Phi^c}_d(y)\rangle =$  $= g^2{\delta^a}_d{\delta^c}_b G(x-y)$

The fermion vertex:



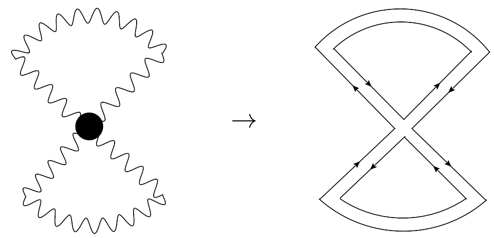
$$= \frac{1}{g^2}{\delta^a}_h{\delta^c}_b{\delta^e}_d{\delta^f}_g$$

So here we can adapt the double line notation:

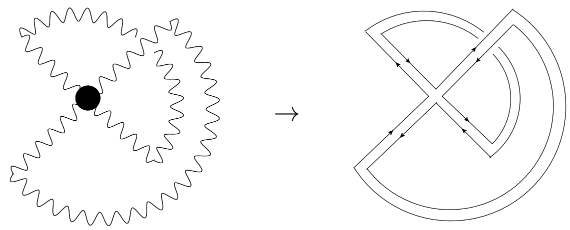


Vacuum energy

We consider vacuum bubbles, *i.e.* diagrams with no external legs. The lowest order diagrams will be



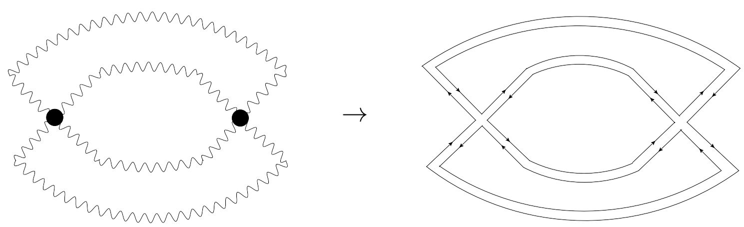
(1)



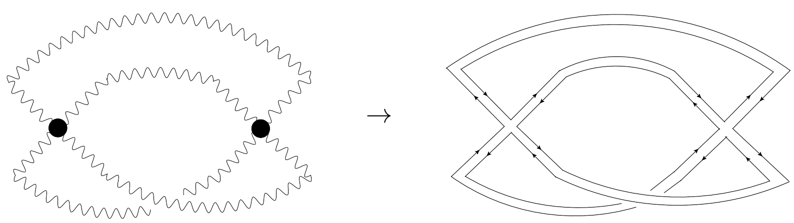
(2)

In the case of diagram 1, each contracted index line gives N, so the total contribution will be of the order $N^3\frac{(g^2)^2}{g^2} = N^3g^2$. In the case of diagram 2, there is only one contracted line, the total contribution will be of the order Ng^2. The difference comes from the fact that the matrices do not commute. In the first case, the diagram can be drawn on a plane without crossing lines, we call it a planner digram; while in the second case, the diagram cannot be drawn on a plane without crossing lines, we call it a non-planar diagram.

If we consider next order in the perturbation theory



(3)

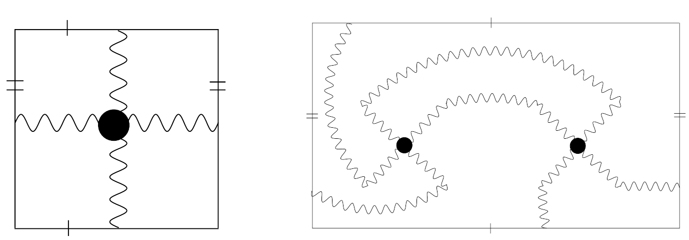


(4)

The first diagram gives the order of N^4g^4, the second diagram gives the order of N^2g^4. We can further consider higher order diagrams, but how can we obtain general N-counting? And how to classify all the non-planar diagrams?

To answer the above questions, we make 2 observations

- Diagrams 2 and 4 can be drawn on a torus without crossing lines.



(5)

- The power of N for each diagram equal to the number of faces in the diagram after we straighten it out.

8.821 / 8.871 String Theory and Holographic Duality
Fall 2014