Physics 8.821: Problem Set 3 Solutions

1. **Conserved world-sheet currents**

   We start with the Polyakov action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_a X^\mu \partial^a X_\mu \ , \tag{1}$$

   where we take the worldsheet coordinates to be (τ, σ). Now if the action is invariant under some global symmetry corresponding to a change of the string coordinates δX^μ, then the usual Noether procedure gives us an expression for the currents

$$\epsilon j_a(\sigma) = \partial_a X^\mu \delta X_\mu, \tag{2}$$

   one for each possible δX_μ, and where the normalization of the currents is arbitrary at the classical level. For the case where the target space is D-dimensional flat spacetime the possible symmetries are those of the Poincare group. These can be divided into two classes, translations and Lorentz transformations.

   For translations we have $\delta X^\mu = \epsilon^\mu$ with a constant vector ϵ^μ, leading to the D conserved currents

$$j^\mu_a = \partial_a X^\mu \tag{3}$$

   It is clear that these correspond to the conservation of four-momentum along the string; in particular, the conserved charges are precisely the total four-momentum P^μ of the string,

$$P^\mu(\tau) \equiv \int d\sigma j_\tau(\sigma, \tau) = \int d\sigma \partial_\tau X^\mu, \tag{4}$$

   and by the usual arguments we have $\frac{d}{d\tau} P^\mu = 0$. Let us turn now to the homogenous part of the Poincare group, Lorentz transformations, where we have $\delta X^\mu = \omega^{\mu\nu} X_\nu$ with an antisymmetric matrix $\omega^{\mu\nu}$ which is a generator of the Lorentz group. The conserved currents are

$$j^B_a = \partial_a X^\mu \omega^B_{\mu\nu} X^\nu, \tag{5}$$

   where B labels the $\frac{D(D-1)}{2}$ generators of the Lorentz group. It seems clear that if $\omega_{\mu\nu}$ is nonzero only along the spatial directions these are just expressions for the angular momentum density of the string, and we will not dwell on this further. Slightly less familiar are the generators of the boosts, i.e. take $\omega_{ti} = -\omega_{it} = 1$. What do these mean?

   This is most transparent in static gauge, in which $X^t = \tau$. In that case we can directly construct the current corresponding to a boost,

$$j^{boost}_\tau = X^i - \partial_\tau X^i \tau \qquad j^{boost}_\sigma = -\partial_\sigma X^i \tag{6}$$

Let us now construct the conserved charge Q^{boost} corresponding to this symmetry,

$$Q^{boost} = \int d\sigma j_\tau^{boost} = \int d\sigma X^i - \tau \int d\sigma \partial_\tau X^i \equiv X_{CM}^i - \tau P^i \tag{7}$$

where in the last equality we have used the expression (4) for the spatial momentum of the string and identified the center of mass position of the string coordinate $X_{CM}^i \equiv \int d\sigma X^i$. The fact that this quantity Q^{boost} is constant for each of the $D-1$ possible boosts thus means simply that the center of mass of the string moves in a straight line through spacetime, which makes perfect sense.

2. **Virasoso algebra**

As this problem is a standard computation treated in all textbooks in string theory (e.g. see Chapters 9 and 12 of [1]) we will be somewhat brief here. We have the usual open-string mode expansion

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma, \tag{8}$$

(note the factor of i different from the problem set – I believe this is necessary) and we work in light-cone gauge $X^+ = 2\alpha' p^+ \tau$.

(a) We can explicitly work out the stress-energy tensor on the world-sheet to be

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\partial_c X^\mu \partial^c X_\mu \ . \tag{9}$$

The trace of this symmetric tensor vanishes identically; thus the condition that it vanish $T_{ab} = 0$ is only two equations, and we can explicitly work out what they are to find the conditions

$$\left(\dot{X} \pm X'\right)^2 = 0, \tag{10}$$

where an overdot $\cdot \equiv \partial_\tau$ and a prime $' \equiv \partial_\sigma$, and the square represents an inner product with respect to the target-space Lorentz metric. Now using the choice of light-cone gauge $X^+ = 2\alpha' p^+ \tau$ we see that this constraint becomes the remarkably simple

$$\left(\dot{X}^- \pm X'^-\right) = \frac{\left(\dot{X}^i \pm X'^i\right)^2}{4\alpha' p^+}, \tag{11}$$

i.e. a *linear* equation for the X^-'s; this is of course the miracle of light-cone gauge that explicitly allows us to solve the constraints. Now using the mode expansion (35) we find

$$\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'} \sum_n \alpha_n^\mu e^{-in(\tau\pm\sigma)}. \tag{12}$$

Plugging this into (11) we find

$$\sum_p \alpha_n^- e^{-in(\tau\pm\sigma)} = \frac{1}{2p^+\sqrt{2\alpha'}} \sum_{n,m} \alpha_n^i \alpha_m^i e^{-i(n+m)(\tau\pm\sigma)}, \tag{13}$$

which permits the immediate solution

$$\alpha_n^- = \frac{1}{2p^+\sqrt{2\alpha'}} \sum_m \alpha_{n-m}^i \alpha_m^i \tag{14}$$

This is fine, except that we would like to express this in terms of the Virasoro generators L_m, which are basically the same as the right-hand side of the above equation (with the p^+ extracted of course) except that they are *normal ordered*:

$$L_n = \frac{1}{2} \sum_m : \alpha_{n-m}^i \alpha_m^i : \tag{15}$$

Thus to relate (14) to (15) we may need to commute the α_m^i's on the right-hand side of of (14) past each other, so that all of the α_m with $m > 0$ stand on the *right*. Their commutation relations are

$$[\alpha_m^i, \alpha_n^i] = m\delta_{m+n,0}\delta^{ij} \tag{16}$$

Thus we see that for any of the L_n's with $n \neq 0$ we need not worry, as all of the α_n^i's appearing on the right-hand side will always commute and we can push them around with impunity. However for $n = 0$ we have an issue; we have to commute half of the α_m^i's in in the infinite sum in (14) past each other. Using (19) we see that we will actually thus encounter the sum $\frac{(D-2)}{2}\sum_{m=1}^{\infty} m$. For now we stuff this infinite sum into a factor a, and we then find that

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}(L_n - a\delta_{n,0}) \tag{17}$$

It was discussed in class that a careful treatment of this sum gives $a = \frac{D-2}{24}$.

(b) This is a straightforward but somewhat tedious exercise in keeping track of the limits of various sums. First let us make the normal ordering of the L_n explicit by writing

$$L_n = \frac{1}{2}\left(\sum_{m=-\infty}^{n} \alpha_m^i \alpha_{n-m}^i + \sum_{m=n+1}^{\infty} \alpha_{n-m}^i \alpha_m^i\right) \tag{18}$$

From here it is easy to compute the commutator the L_n with one α_p^j to be

$$[L_n, \alpha_p^j] = -p\alpha_{n+p}^j \tag{19}$$

This is a useful auxiliary result. We are now ready to compute the commutator of L_m with L_n. We write

$$[L_m, L_n] = \frac{1}{2}\left[L_m, \sum_{p=-\infty}^{n} \alpha^i_p \alpha^i_{n-p} + \sum_{p=n+1}^{\infty} \alpha^i_{n-p}\alpha^i_p\right] \tag{20}$$

Now using (19) we find the answer to be the two sums

$$\begin{aligned}[L_m, L_n] = & -\frac{1}{2}\sum_{p=-\infty}^{n}\left[(n-p)\alpha^i_p\alpha^i_{m+n-p} + p\alpha^i_{p+m}\alpha^i_{n-p}\right] \\ & -\frac{1}{2}\sum_{p=n+1}^{\infty}\left[(n-p)\alpha^i_{n-p+m}\alpha^i_p + p\alpha^i_{n-p}\alpha^i_{m+p}\right]\end{aligned} \tag{21}$$

Note the structure of these two sums; the second line is identical to the first except that the limit on the sum is different and the ordering of the α's is reversed.

Let us now handle two possible cases separately. First, consider the case $m \neq -n$. In that case staring at the α products we see that they always commute, and the two lines are thus *identical* except for the limit on the sums. We may then combine the two sums into one and shift $p \to p - m$ on the second part of each sum; the piece proportional to p cancels and we then find the full expression

$$[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{\infty}(m-n)\alpha^i_p\alpha^i_{m+n-p} = (m-n)L_{m+n} \qquad (m \neq -n) \tag{22}$$

Now, on to the trickier case when $m = -n$. The expression then takes the form

$$\begin{aligned}[L_{-n}, L_n] = & -\frac{1}{2}\sum_{p=-\infty}^{n}\left[(n-p)\alpha^i_p\alpha^i_{-p} + p\alpha^i_{p-n}\alpha^i_{n-p}\right] \\ & -\frac{1}{2}\sum_{p=n+1}^{\infty}\left[(n-p)\alpha^i_{-p}\alpha^i_p + p\alpha^i_{n-p}\alpha^i_{p-n}\right]\end{aligned} \tag{23}$$

To write this whole expression in terms of L_0 we essentially need to normal-order the expression; this will involve extracting out a c-number contribution from the commutators of the oscillators. To understand how to do this, imagine that $n > 0$. Then every term is already normal-ordered *except* for a part of the first term, which must be broken into two parts:

$$\sum_{p=-\infty}^{n}(n-p)\alpha^i_p\alpha^i_{-p} = \sum_{p=-\infty}^{0}(n-p)\alpha^i_p\alpha^i_{-p} + \sum_{p=1}^{n}(n-p)\left(\alpha^i_{-p}\alpha^i_p + p(D-2)\right) \tag{24}$$

The last term is the c-number contribution from normal-ordering the $\alpha's$; evaluating it we find

$$\sum_{p=1}^{n}(n-p)p = \frac{n^3-n}{6} \tag{25}$$

The remainder of the calculation proceeds as above. By shifting the range of the sum $p \to p - n$ in two of the sums and combining terms the expression can be written as

$$[L_{-n}, L_n] = -\frac{1}{2}\left[\sum_{p=-\infty}^{0} 2n\alpha^i_p\alpha^i_{-p} + \sum_{p=1}^{\infty} 2n\alpha^i_{-p}\alpha^i_p\right] - \frac{D-2}{12}(n^3 - n) \tag{26}$$

From (18) we see that the operator part of this expression is just $-2nL_0$; thus this expression fits into the same form as (22), except with an extra c-number contribution. Combining this answer with the other case (22) we can thus write the whole answer as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0}, \tag{27}$$

which is indeed the desired result.

(c) Before working out the full expression, we use the fact that $\alpha^i_0 = \sqrt{2\alpha'}p^i$ to work out the transformation of the zero mode of the string under the Virasoro algebra,

$$[L_n, x^i] = -i\alpha^j_n\sqrt{2\alpha'} \tag{28}$$

Now using this together with (19) and the mode expansion (35) it is easy to calculate

$$[L_n, X^i] = \sqrt{2\alpha'}\left(-i\alpha^i_n - i\sum_{m\neq 0}\alpha^i_{m+n}e^{-im\tau}\cos(m\sigma)\right), \tag{29}$$

where the first term comes from the zero mode. We may rewrite this as one sum

$$[L_n, X^i] = -i\sqrt{2\alpha'}\left(\sum_m \alpha^i_{m+n}\frac{1}{2}\left(e^{-im\sigma^+} + e^{-im\sigma^-}\right)\right) \tag{30}$$

Now we should work out the transformation of $X^i(\sigma^\pm)$ under the diffeomorphisms generated by $\xi^\pm_n \equiv -ie^{in\sigma^\pm}$. Under one such diffeomorphism we have

$$\delta_n X^i = \epsilon\left(\xi^+_n\partial_+X^i + \xi^-_n\partial_-X^i\right) \tag{31}$$

which works out to be

$$\delta_n X^i = -i\epsilon\sqrt{2\alpha'}\left(\alpha_0\frac{1}{2}\left(e^{in\sigma^+} + e^{-in\sigma^-}\right) + \sum_{m\neq 0}\alpha^i_m\frac{1}{2}\left(e^{-i(m-n)\sigma^-} + e^{-i(m-n)\sigma^+}\right)\right) \tag{32}$$

Again the first term comes from the transformation of the zero mode, and again this may be written as a single sum; after doing this and shifting $m \to m+n$ we see that the expression we get is identical to (30). We conclude that

$$\epsilon[L_n, X^i] = \delta_n X^i, \tag{33}$$

i.e. the Virasoro algebra generated diffeomorphisms on the worldsheet.

(d) There are various ways to understand this. One way is to note that in light-cone gauge we have truly reduced our state space to a set of $(D-2)$ transverse oscillators; the equation of motion for each of these transverse modes is

$$\ddot{X}^i - X''^i = 0, \tag{34}$$

This is precisely the equations of motion for a CFT in two dimensions, the CFT of $D-2$ massless bosons. The Virasoro algebra that we have constructed is actually the mode expansion for the stress tensor T_{ab} of this CFT. This CFT should have central charge $D-2$, and indeed we have checked that it does in in (27). The modes of the stress tensor should generate conformal transformations on the operators of the theory, and indeed that is precisely what the Virasoro operators do, as shown explicitly in (33) above.

Note that we actually do not obtain the *full* possible conformal transformations of a 2d field theory, as we must transform σ^+ and σ^- in the same way; this is because we are studying the open string, and the boundary conditions at the open string endpoints break the full conformal group (i.e. left and right-moving) down to a diagonal subgroup that preserves these boundary conditions.

3. **Scalar modes as transverse fluctuations of D-branes**

As this is not a problem we do not actually present a solution here.

4. **Separation of D-branes as Higgs mechanism**

This problem is worked out in detail in Section 14.3 of [1]. In our treatment we will let i run over all of the transverse directions that are not X^{25}.

(a) It is clear that the mode expansions for all directions tangent to the brane (i.e. the X^i) are not affected by its presence, i.e. for those we have the usual

$$X^i = x^i + 2\alpha' p^i \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^i}{n} e^{-in\tau} \cos n\sigma, \tag{35}$$

The direction X^{25} should be different; if we work in a gauge where the worldsheet is a string with length π, then it is clear from inspection that the mode expansion must be

$$X^{25} = a\frac{\sigma}{\pi} + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^{25}}{n} e^{-in\tau} \sin n\sigma \tag{36}$$

To see that this is correct, note that the string should have no net momentum in the X^{25} direction, thus there is no $p^{25}\tau$ term; however $X^{25}(\sigma = 0)$ and $X^{25}(\sigma = \pi)$ must differ by a for all time, thus there is a linear term in σ and the spatial mode expansion now has $\sin n\sigma$ rather than $\cos n\sigma$.

It is convenient for what follows to use the notation $\sqrt{2\alpha'}\alpha_0^{25} = \frac{a}{\pi}$; in that case the derivative expressions (12) calculated previously still hold.

(b) To quantize the string we use the standard mass-shell condition, discussed in class

$$2p^+p^- = \frac{1}{\alpha'}\left(\alpha' p^i p^i + \frac{1}{2}\alpha_0^{25}\alpha_0^{25} + \sum_{n=1}^{\infty}\left[\alpha_{-n}^i\alpha_n^i + \alpha_{-n}^{25}\alpha_n^{25}\right] - 1\right) \tag{37}$$

This expression does not depend on boundary conditions, although we have anticipated what comes next by separating out the X^{25} direction. Now the momentum along the X^i directions should be interpreted as the mass-square of the particles; we have then

$$M^2 = 2p^+p^- - p^ip^i = \left(\frac{a}{2\alpha'\pi}\right)^2 + \frac{1}{\alpha'}(N-1) \tag{38}$$

where we have used the expression for α_0^{25} argued above and the level N has its usual definition

$$N = \sum_{n=1}^{\infty}\left(\alpha_{-n}^i\alpha_n^i + \alpha_{-n}^{25}\alpha_n^{25}\right) \tag{39}$$

Thus depending on the value of a there may still be a tachyon if $N = 0$; more interesting, however, are the states at $N = 1$. These have 25-dimensional mass-squared equal to

$$M^2 = \left(\frac{a}{2\alpha'\pi}\right)^2 \tag{40}$$

and can be found by either exciting either α_{-1}^i in which case we have $25 - 2 = 23$ massive states that are in a vector representation of the $SO(24,1)$ Lorentz group *or* the α_{-1}^{25}, which naively appears to be a massive scalar under $SO(24,1)$.

However, we know that a massive vector in 25 spacetime dimensions must have 24 states; thus it seems that the scalar is really a part of the vector states and we should think of the spectrum as simply being the 24 states corresponding to a massive vector living in the 25 dimensions parallel to the two D-branes.

Note that there is also a string stretching between the two branes with the other orientation; thus the spectrum of massive states above is actually doubled.

(c) First let us finish our counting of string states above. Denoting by [12] and [21] the strings stretching between the two branes, we conclude that at $N = 1$ for both [12] and [21] we have 24 states that are probably a massive vector. Similarly we have [11] and [22] strings that do not stretch between the branes; for each of these we have a massless scalar (1 state) and a massless vector (23 states).

Now we turn to the field theoretical problem. The Lagrangian is

$$S = \int d^{25}x \; \mathrm{Tr}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}(D_\mu\phi)(D^\mu\phi)\right) \tag{41}$$

Let us parametrize the $U(2)$ gauge field as follows:

$$A_\mu = A_\mu^0\frac{\mathbf{1}}{2} + A_\mu^i\frac{\sigma_i}{2} \tag{42}$$

Thus A^0 is the diagonal $U(1)$. Now we want to study the spectrum in the Higgs phase where the scalar ϕ has expectation value

$$\phi_0 = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} \qquad v = \frac{a}{2\pi\alpha'} \tag{43}$$

What is the spectrum? The scalar has no potential and is unaffected by the shift in the vacuum; thus naively we have 4 massless scalar states. To determine the mass terms for the gauge field, we work out the contribution from the kinetic term of the scalar. Using the definition of the adjoint covariant derivative

$$D_\mu \phi = \partial_\mu \phi - i[A_\mu, \phi], \tag{44}$$

we find the contribution from the background (43) to the gauge field to be

$$\frac{1}{2}(D_\mu \phi_0)(D^\mu \phi_0) = \frac{v^2}{2}((A^1)^2 + (A^2)^2) \tag{45}$$

Thus two of the gauge fields have received mass v. Indeed, it is clear that (43) transforms under two of the generators of $U(2)$ and has thus given the corresponding gauge bosons mass. This means that of the four massless scalars two of them must be Goldstone bosons that are eaten by the gauge fields; so our final count gives us two massive vectors (with mass $\frac{a}{2\pi\alpha'}$), two massless vectors, and the two remaining massless scalars, in agreement with the string-theoretical calculation above.

(d) We now have N Dp-branes, sitting at positions

$$\vec{X}^{(k)} = \left(a_1^{(k)}, \cdots, a_{25-p}^{(k)}\right) \qquad k = 1, 2, \cdots, N \tag{46}$$

What is the relevant scalar expectation value? We now have $25 - p$ scalar fields ϕ_i, each of which is an $N \times N$ matrix; with some thought it is possible to convince yourself that the answer is

$$\phi_i = \frac{1}{2\pi\alpha'} \begin{pmatrix} \alpha_i^{(1)} & 0 & & \\ 0 & \alpha_i^{(2)} & 0 & \\ & 0 & \cdots & 0 \\ & & 0 & a_i^{(N)} \end{pmatrix}, \tag{47}$$

i.e. a diagonal matrix with the coordinates of the transverse separation of the Dp-branes along the diagonals. One can check by explicitly constructing commutators that this construction will give masses

$$M_{ab}^2 = \frac{(\vec{X}^a - \vec{X}^b)^2}{(2\pi\alpha')^2} \tag{48}$$

to the off-diagonal gauge bosons A_{ab}, as desired.

5. **Mass of a D-brane**

(a) We have discussed in class that a string diagram with an Euler characteristic of χ, where

$$\chi = 2 - 2 \times \text{holes} - \text{boundaries} \tag{49}$$

corresponds to an amplitude of $g_s^{-\chi}$. If we want to sum over vacuum diagrams of open strings, the leading order contribution is simply the disc, which has zero holes and one boundary, and thus $\chi_{disc} = 1$. The mass of the D-brane should thus scale like the relevant vacuum diagram, which gives us

$$M \sim \frac{1}{g_s}. \tag{50}$$

(b) It was discussed in class that Newton's constant $G_N \sim g_s^2$. Suppose we have D bulk dimensions (where D is either 10 or 26 depending on whether we are doing superstring theory or bosonic string theory.) Then $G_N \sim g_s^2 l_s^{2-D}$. The interaction energy E between two Dp branes separated by a transverse distance R is then

$$E \sim \frac{G_N M^2}{R^{D-3-p}} \sim (g_s)^0 \tag{51}$$

One could also arrive at this scaling by considering the stringy amplitude for graviton exchange between two Dp-branes; at lowest level the diagram has the topology of a cylinder (see e.g. p274 of [2]) and thus has Euler number 0.

This should be compared to their mass, which is $2M \sim g_s^{-1}$ and is thus always larger.

(c) The gravitation potential due to a single D-brane is proportional to $G_N M \sim g_s$ which can neglected in the regime $g_s \to 0$. If we consider N D-branes, then the gravitational potential scales as

$$G_N N M \sim g_s N \tag{52}$$

which can no longer be ignored if N is large enough so that

$$g_s N \sim 1 \tag{53}$$

We note that in the superstring context the force between parallel D-branes is actually exactly 0 due to an intricate cancellation between the form-charges that the D-branes carry and their gravitational attraction; nevertheless they do distort the geometry around them, and when $g_s N \gg 1$ they do so in a way that we can understand, causing all of the delightful effects that we now call AdS/CFT.

References

[1] B. Zwiebach, "A First Course in String Theory," Cambridge University Press, Cambridge (2004).

[2] J. Polchinski, "String Theory: Volume I," Cambridge University Press, New York 2001.

8.821 / 8.871 String Theory and Holographic Duality
Spring 2015