Physics 8.821: Problem Set 1 Solutions

1. **Matrix element identities in the proof of Weinberg-Witten Theorem**

   Here we seek to establish various identities related to states in a relativistic quantum field theory. We are considering single-particle states $|k, \sigma\rangle$ with $k = k^\mu = (k^0, \vec{k})$ the four-momentum and σ an internal quantum number of the particle. The states are normalized as

   $$\langle k, \sigma | k', \sigma' \rangle = \delta_{\sigma, \sigma'} \delta^{(3)}(\vec{k} - \vec{k}') \; . \tag{1}$$

   These states have charge q under a $U(1)$ current J^μ.

   We tackle the current problem first. By the definition of charge, we have

   $$Q|k', \sigma'\rangle = \int d^3x \, J^0(\vec{x})|k', \sigma'\rangle = q|k', \sigma'\rangle \; . \tag{2}$$

   Note we can write $J^0(\vec{x}) = e^{i\vec{P}\cdot\vec{x}} J^0(0) e^{-i\vec{P}\cdot\vec{x}}$ where here P represents the momentum *operator*. Hitting this with $\langle k, \sigma|$ and using the fact that $J^0(\vec{x})$ is sandwiched between two momentum eigenstates we find

   $$\int d^3x \, e^{i\vec{x}\cdot(\vec{k}-\vec{k}')} \langle k, \sigma | J^0(0)|k', \sigma'\rangle = q\delta_{\sigma, \sigma'}\delta^{(3)}(\vec{k} - \vec{k}') \; . \tag{3}$$

   The integral over x on the left-hand side gives us a $(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$, and we conclude that

   $$\lim_{k \to k'} \langle k, \sigma | J^0 | k', \sigma' \rangle = \frac{q}{(2\pi)^3} \delta_{\sigma, \sigma'} \tag{4}$$

   The limit $k \to k'$ is necessary as the fact that everything is multiplied by a delta function means that the equations vanish identically away from $k \to k'$, giving us no information.

   We are not done yet. We now seek to figure out the action of the whole current J^μ, where μ can be any index. We will do this by studying the Lorentz transformation properties of various quantities in this expression. First, recall from [1] that given a Lorentz transformation Λ, it is represented on the Hilbert space by a unitary operator $U(\Lambda)$, whose action on a state $|k, \sigma\rangle$ is written

   $$U(\Lambda)|k, \sigma\rangle = \left(\frac{N(k)}{N(\Lambda k)} \right) \sum_{\bar{\sigma}} D_{\bar{\sigma}\sigma}(\Lambda)|\Lambda k, \bar{\sigma}\rangle, \tag{5}$$

   where $D_{\bar{\sigma}\sigma}(\Lambda)$ is a set of numbers, forming a representation of the little group that will not much concern us. $N(k)$ is however an important normalization, and it is shown in (2.5.18) of [1] that it is given by

   $$N(k) = \sqrt{\frac{r^0}{k^0}}, \tag{6}$$

   where r^μ is a "reference four-vector" whose exact details will again not concern us. Now take some Λ and consider the expression

   $$\lim_{k \to k'} \langle k, \sigma | U^\dagger(\Lambda) J^0 U(\Lambda) | k', \sigma' \rangle \tag{7}$$

Via the Lorentz transformation of J^μ, this is equal to

$$\lim_{k \to k'} \langle k, \sigma | \Lambda^0{}_\mu J^\mu | k', \sigma' \rangle . \tag{8}$$

On the other hand, we can also act with the U's on the *states*, finding

$$\lim_{k \to k'} \sqrt{\frac{\Lambda^0{}_\rho k^\rho \Lambda^0{}_\mu k'^\mu}{k^0 k'^0}} \sum_{\bar\sigma' \bar\sigma} \langle \Lambda k, \bar\sigma | D_{\bar\sigma\sigma}(\Lambda)^\dagger J^0 D_{\bar\sigma'\sigma'}(\Lambda) | \Lambda k', \bar\sigma' \rangle . \tag{9}$$

Now the $D's$ are just numbers that can be freely moved through the J^0, and we find after using (4), remembering that D's should be unitary, and equating the result to (8):

$$\lim_{k \to k'} \langle k, \sigma | \Lambda^0{}_\rho J^\rho | k', \sigma' \rangle = \frac{\Lambda^0{}_\rho k^\rho}{k^0} \frac{q}{(2\pi)^3} \delta_{\sigma,\sigma'} \tag{10}$$

As this must hold for *all* Lorentz transformations Λ we realize that the only way it can be true is if

$$\lim_{k \to k'} \langle k, \sigma | J^\rho | k', \sigma' \rangle = \frac{q}{(2\pi)^3} \frac{k^\rho}{k^0} \delta_{\sigma,\sigma'}, \tag{11}$$

which is indeed the desired result.

As the corresponding calculation for $T^{\mu\nu}$ is basically exactly the same with an extra index I will be brief and just state the analogs of various equations for the $U(1)$ case. The analog of charge is now total momentum $P^\mu = \int d^3x \, T^{0\mu}$. (4) becomes

$$\lim_{k \to k'} \langle k, \sigma | T^{0\nu} | k', \sigma' \rangle = \frac{1}{(2\pi)^3} \delta_{\sigma,\sigma'} k^\mu \tag{12}$$

Arguments identical to those above lead us to the analog of (10)

$$\lim_{k \to k'} \langle k, \sigma | \Lambda^\mu{}_\rho \Lambda^0{}_\alpha T^{\rho\alpha} | k', \sigma' \rangle = \frac{\Lambda^\mu{}_\rho \Lambda^0{}_\alpha k^\rho k^\alpha}{(2\pi)^3 k^0} \delta_{\sigma,\sigma'}, \tag{13}$$

which again leads us to conclude that

$$\lim_{k \to k'} \langle k, \sigma | T^{\mu\nu} | k', \sigma' \rangle = \frac{1}{(2\pi)^3} \frac{k^\mu k^\nu}{k^0} \delta_{\sigma,\sigma'} . \tag{14}$$

2. $SU(2)$ **Yang-Mills theory as a** $U(1)$ **theory**

We are studying an $SU(2)$ gauge theory with the normal action

$$S = -\int d^4x \frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a}, \tag{15}$$

and we are trying to understand the statement that apparently this theory does *not* have a "Lorentz-covariant conserved four-vector current corresponding to the $U(1)$ symmetry", where the $U(1)$ is the subgroup of $SU(2)$ generated by σ^3.

Let us begin by constructing the conserved currents. We will use the normal Noether procedure, which we recall briefly: note that the existence of a conserved current is actually related

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to the existence of a *global* symmetry, i.e. given an action that is invariant under a global transformation

$$\delta\Phi = \epsilon T\Phi \tag{16}$$

where T is a generator of some sort, one promotes this global transformation to a spacetime-dependent one

$$\delta\Phi = \epsilon(x)T\Phi, \tag{17}$$

and then finds that due to the spatial dependence of the transformation the action is *not* invariant, i.e. $\delta S = \int d^4x j^\mu \partial_\mu \epsilon$. One then reads off j^μ and concludes that it is conserved on-shell, as on-shell even this variation must vanish (since indeed *all* variations vanish).

We now apply this in our context; the global gauge rotations take the form

$$\delta A_\mu^a = f^{abc} A_\mu^b \Lambda^c, \tag{18}$$

where Λ^c is a constant gauge parameter and f^{abc} are the structure constants of the group. We now promote this to a spacetime-dependent transformation

$$\delta A_\mu^a = f^{abc} A_\mu^b \Lambda^c(x) \ . \tag{19}$$

Note that we are *not* performing a local gauge transformation – while it might look similar, that is something philosophically *different*, involving an extra $\partial_\mu \Lambda^a$ term. If we performed a local gauge transformation we would of course find that the action is completely invariant. But then of course the Noether procedure would fail. We are doing something simpler – all we are doing is finding the conserved current associated with the *global* subgroup of the gauge symmetry, and we are following the standard algorithm to do it.

We now compute the variation of the action under this transformation, which turns out to be

$$\delta S = -\frac{1}{g^2} \int d^4x F_a^{\mu\nu} f^{abc} A_\nu^b \partial_\mu \Lambda^c \tag{20}$$

This is quite straightforward, as the only terms we are interested in are those which are proportional to $\partial_\mu \Lambda^c$, since we know all the rest vanish anyway. Thus we read off the current (for a general non-Abelian theory) as

$$j_c^\mu = F_a^{\mu\nu} f^{abc} A_\nu^b, \tag{21}$$

where I have dropped some overall factors since they won't concern us. By construction, this current *must* be conserved. Now for the purposes of the problem set we specialize to $SU(2)$, for which we have $f^{abc} = \epsilon^{abc}$ and we fix $c = 3$ in the current (21). It is convenient to work with the linear combinations

$$B_\mu^\pm = \frac{1}{\sqrt{2}} \left(A_\mu^1 \pm i A_\mu^2 \right) \tag{22}$$

and the similarly defined $F_{\mu\nu}^\pm$. (21) in this case then becomes

$$j_3^\mu = i \left(F_+^{\mu\nu} B_\nu^- - F_-^{\mu\nu} B_\nu^+ \right), \tag{23}$$

where $F_{\mu\nu}^\pm$ is

$$F_{\mu\nu}^\pm = D_\mu B_\nu^\pm - D_\nu B_\mu^\pm, \tag{24}$$

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and where the covariant derivative D_μ of $B_\nu^\pm$ that is used in this formula reflects the fact that the fields defined as (22) are charged under the $U(1)$ generated by A^3, i.e.

$$D_\mu B_\nu^\pm \equiv \left(\partial_\mu \pm i A_\mu^3\right) B_\nu^\pm \tag{25}$$

All of this is repackaging of the expression (21).

Now we turn to the real problem; the expression (21) or the $SU(2)$ specialization (23) are conserved by construction – but are they Lorentz-invariant? The real answer is *no*, and it is because *massless vector fields transform strangely under Lorentz transformations*. If you are not familiar with this it may be a surprise, and I refer you to [1] Section 5.9 for an authoritative (or [3] for a more informal) discussion. The key fact is that under the unitary operator $U(\Lambda)$ representing a Lorentz transformation Λ the local quantum operator corresponding to a massless vector field B_μ transforms as

$$U(\Lambda)B_\mu(x)U^\dagger(\Lambda) = \Lambda^\nu{}_\mu B_\nu(\Lambda x) + \partial_\mu \Omega(x, \Lambda) \tag{26}$$

There is an extra inhomogenous piece proportional to the gradient of a scalar function $\Omega(x, \Lambda)$ which depends on the parameters of the Lorentz transformation.

We now pause to describe how something like this can be. The representation theory of quantum fields tells us that the possible states of a particle are governed by representations of the "little group", the set of Lorentz transformations that leaves a reference momentum k^μ (which is lightlike in the massless vector case) invariant. For a massless vector we expect the little group to contain an $SO(2)$ rotation subgroup around the spatial momentum direction; the polarization ϵ^ν with definite eigenvalues under this rotation form the two helicities, which is all we should have for a massless vector, i.e. parametrizing the rotation by θ:

$$(D_{SO(2)}(\theta)\epsilon_\pm(k))^\nu = \exp(\pm i\theta)\epsilon_\pm^\nu(k) \tag{27}$$

However it turns out that the little group is larger than this Abelian group; it also contains an awkward non-compact factor, and when acting on the states of definite helicity it generally brings in an admixture of a *longitudinal* polarization, i.e.

$$(D_{full}(c)\epsilon_\pm(k))^\nu = \epsilon_\pm^\nu(k) + c\frac{k^\mu}{|\vec{k}|} \tag{28}$$

This is the term proportional to $\partial_\mu \Omega$ seen above.

This suggests that in some fundamental sense *massless vector fields have some problem with quantum mechanics*. There is one way to save this and recover Lorentz-invariance – let us *demand* that all couplings of the vector field B_μ are invariant under the shift $B_\mu \to B_\mu + \partial_\mu \Omega$. In that case this anomalous Lorentz transformation will not manifest itself in physical amplitudes; we recognize this now as the statement that "massless vector fields have gauge invariant actions, etc."

We now return to our current (21). The key point is that it depends explicitly on the massless gauge field A_μ^a in a non-gauge-invariant way and so is not invariant under the weird transformation (26), and so does not actually form a Lorentz-covariant four-vector. If we could have written down a *gauge-invariant* conserved current it would indeed be a good four-vector; I don't know of a way to prove that this is impossible but leave it to you to convince yourself of it.

3. **Falling into a black hole!**

This is a standard calculation from general relativity courses and I recommend [2], Section 5.6 for more discussion.

Recall that the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2 \tag{29}$$

We can restore the G_N by taking $M \to MG_N$.

(a) We will use an overdot to represent differentiation with respect to proper time τ, i.e. $\cdot \equiv \frac{d}{d\tau}$. To solve the geodesic equation, we use the fact that the four-velocity of the observer is always -1,

$$-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{1}{1 - \frac{2M}{r}}\dot{r}^2 = -1, \tag{30}$$

and the fact that the existence of the time translation Killing vector means that there is a quantity E conserved along the geodesic,

$$\left(1 - \frac{2M}{r}\right)\dot{t} = E . \tag{31}$$

These two equations let us reduce the problem to just the radial motion,

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right) = E^2 . \tag{32}$$

This equation can actually be integrated explicitly but we will not do this. Instead let us look at the incremental change in proper time $d\tau$ for a small change in radius dr; from (32) we have

$$d\tau = \frac{dr}{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)}} \tag{33}$$

Near the horizon at $r = 2M$ (which is the only point where anything strange could happen) we see that

$$d\tau = \frac{dr}{E} \qquad (r \to 2M) \tag{34}$$

which is perfectly well-behaved and regular; thus we can presumably move all the way through up to $r = 2M$ in finite proper time. The same result can be reached from a direct integration of (32).

On the other hand, combining (31) with (34) we find

$$dt = \frac{1}{1 - \frac{2M}{r}}dr \qquad (r \to 2M), \tag{35}$$

Upon integration we see that the change in t is logarithmically divergent as we approach $r \to 2M$; thus the observer takes infinite coordinate time t to reach the horizon.

(b) The easiest way to see this is to convert to Eddington-Finkelstein coordinates, given by

$$v = t + r_* \qquad r_* = r + 2M \log\left(\frac{r}{2M} - 1\right) . \tag{36}$$

In these coordinates the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 . \tag{37}$$

From here it is immediately clear that the line $v = $ const is a null trajectory; as r_* is a monotonically increasing function of r it is an *infalling* null trajectory. To find the equation for the outgoing trajectory we set $ds^2 = 0$ to find

$$\frac{dv}{dr} = \begin{cases} 0 & \text{infalling} \\ 2\left(1 - \frac{2M}{r}\right)^{-1} & \text{outgoing} \end{cases} \tag{38}$$

Note what is happening; the infalling geodesic is always a line of constant v, but the outgoing trajectory is changing radial direction as we cross $r = 2M$. For $r < 2M$ we see that $\frac{dr}{dv} < 0$ for all light rays. Thus light rays that move forwards in time (i.e. increasing v) necessarily move inwards in radial coordinate. Thus there is no way for our infalling observer to send a message outside the black hole once he crosses $r = 2M$, but he can still receive ingoing messages.

(c) It is clear from the discussion above that if our hapless observer stays within the light cone, once he crosses the horizon he will always be moving to smaller and smaller r, and thus must eventually hit the singularity at $r = 0$. We compute his proper time elapsed from the horizon to the singularity along the geodesic (33)

$$\Delta\tau = \int_0^{2M} \frac{dr}{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)}} \tag{39}$$

Now it is clear from the expression that this integral will be maximized when $E = 0$; physically this can be seen by demanding that the observer is at rest $\dot{r} = 0$ at the horizon and using (30) together with (31). To find the maximal proper time we should thus set $E = 0$. By defining a dimensionless variable $x \equiv \frac{r}{2M}$ we can then scale out all factors of $2M$ to find

$$\Delta\tau = 2M \int_0^1 \frac{dx}{\sqrt{\frac{1}{x} - 1}} = M\pi \tag{40}$$

Restoring the G_N we find $\pi G_N M$ as claimed in the problem set. The integral can actually be done for any E but is a more complicated function.

4. **Black hole evaporation**

(a) The main point of this problem is to figure out how much entropy is carried away by the radiation. We do this first through macrosopic thermodynamic arguments. Let us begin by quickly deriving the relevant equations of state for massless particles. As they are scale invariant, it seems that their energy density ϵ at a temperature T must be

$$\epsilon = cT^4 \tag{41}$$

with c some dimensionless constant. Now we want to find the entropy density; from the first law of thermodynamics $d\epsilon = Tds$ we can now compute the entropy density to be

$$s = \frac{4}{3}\frac{\epsilon}{T} \ . \tag{42}$$

Now let us turn to our problem. We have a black hole with mass M and a Hawking temperature T_H. Imagine standing a little bit away from the black hole so that we do not get into any confusion about curvature, etc. Let the black hole radiate away some energy dE. By the assumption that the radiation is perfect blackbody we find that from (42) that the entropy dS_{rad} carried away in the radiation is

$$dS_{rad} = \frac{4}{3}\frac{dE}{T_H} \tag{43}$$

(If this bothers you imagine considering a shell of radiation with some energy at some temperature, etc. etc. Note that (43) is *not* any version of any law of thermodynamics; rather it is the entropy carried by a volume of radiation with total energy dE.)

On the other hand, the black hole has lost energy dE. Now by *explicit gravity calculation* we know from the first law of black hole mechanics that the black hole loses entropy

$$dS_{BH} = \frac{dE}{T_H} \tag{44}$$

No assumption about any sort of equilibrium has been made here; this is a *fact* about black holes (although it does lead you to believe that through this process the black hole remains in thermal equilibrium). We then conclude that

$$dS_{rad} - dS_{BH} = \frac{dE}{T_H}\left(\frac{4}{3} - 1\right) > 0 \tag{45}$$

We are done. Note that the number of species did not play a role in this argument.

We now present a more microscopic derivation of this fact. Consider the photon modes outside the black hole; they may all be treated as simple harmonic oscillators with frequencies $\omega_k = |\vec{k}|$ that are in thermal equilibrium at a temperature $\beta = \frac{1}{T_H}$. Using basic quantum mechanics we may now figure out the partition function, mean energy, etc. for each mode. We find for each mode ω the partition function

$$Z = \mathrm{tr}\left(\exp(-\beta H)\right) = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}, \tag{46}$$

and the mean energy

$$\langle H \rangle = -\frac{\partial}{\partial\beta}\log Z = \frac{\omega}{2}\frac{1}{\tanh\left(\frac{\beta\omega}{2}\right)}, \tag{47}$$

and finally the entropy

$$S = \log Z + \beta\langle H \rangle = -\log\left(1 - \exp^{-\beta\omega}\right) + \beta\left(\langle H \rangle - \frac{\omega}{2}\right) \tag{48}$$

It seems reasonable to identify $dE = \langle H \rangle - \frac{\omega}{2}$ as the energy carried away by the mode; note that we separate out the zero point energy, as it is always present and so cannot be involved in any sort of energy transfer. We then see that the entropy in this mode dS is always greater than $\frac{dE}{T_H}$, as the argument of the logarithm is always less than 1. Note that if we were to integrate these oscillator expressions over all k it would amount to *deriving* blackbody radiation and we should be able to recover the factor of $\frac{4}{3}$ above.

(b) Let us attempt to keep all the units. If the black hole has mass M, we have for the horizon radius and temperature

$$r_s = \frac{2G_N M}{c^2} \qquad T_H = \frac{\hbar c^3}{8\pi G_N M k_B} \tag{49}$$

Now we assume that the black hole radiates through the blackbody Stefan-Boltzmann expression, i.e. the power is

$$P = \frac{n\sigma}{2} T_H^4 A_{BH} \tag{50}$$

with n the number of massless polarizations, σ is the Stefan-Boltzmann constant, $\sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2}$, A_{BH} is the area of the horizon, and the 2 is because the expression written down for σ assumes that $n = 2$, corresponding to physical photon states in four dimensions. We now write $P = -\frac{dE}{dt}$ with $E = Mc^2$ to find a differential equation for M,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2} \qquad \alpha = \frac{n}{2} \frac{4\hbar c^4}{15 \cdot 8^4 \pi^2 G_N^2} \tag{51}$$

Integrating this from an initial mass M_0 to 0 we find that the the black hole evaporates in a total time given by

$$T = \frac{M_0^3}{3\alpha} = M_0^3 \left(\frac{G_N^2 \pi}{\hbar c^4} \frac{10240}{n} \right) . \tag{52}$$

References

[1] S. Weinberg, "The Quantum Theory of Fields, Volume I," Cambridge University Press, New York 1995.

[2] S. M. Carroll, "An Introduction to General Relativity: Spacetime and Geometry," Addison Wesley, San Francisco 2004.

[3] A. Jenkins, "Topics in particle physics and cosmology beyond the standard model," arXiv:hep-th/0607239.

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Spring 2015