Determination of G from $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e \, (+\gamma)$

$$\Gamma_\mu(\text{experimental}) = \frac{G^2 m_\mu^5}{192\pi^3}$$

correction factors due to:

QED $\quad * \left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\left(1 + \frac{2\alpha}{3\pi} \ln \frac{m_\mu}{m_e}\right)\right]$

Weak $\quad * \left[1 + \frac{3}{5}\frac{m_\mu^2}{m_W^2}\right]$

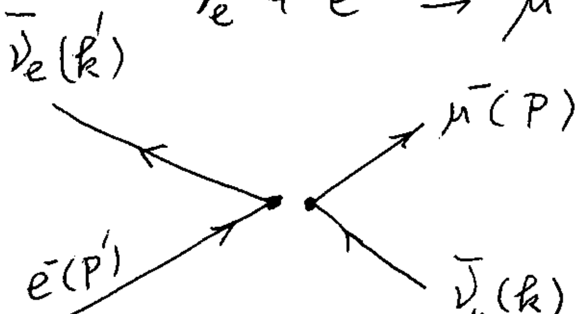
phase space ($\because m_e \neq 0$) $\quad * \left[1 - 8\frac{m_e^2}{m_\mu^2}\right]$

$\frac{\Delta\Gamma_\mu}{\Gamma_\mu} \sim 4.2 \times 10^{-3}$ correction! $\quad \Gamma_\mu(\text{exp.}) = (2.19703 \pm 0.00004) \times 10^{-6}\,\text{s}$

Only then, you get $\quad G(\text{Fermi}) = 1.16639 \pm 0.00002 \times 10^{-5}/\text{GeV}^2$

High energy consequence of Fermi Weak Interaction

$$\bar{\nu}_e + e^- \to \mu^- + \bar{\nu}_\mu$$



$$\Sigma|M|^2 = 128\, G^2 (p' \cdot k)(P \cdot k')$$

$$= 8 G^2 s\, (1 + \cos\theta)^2$$

$$\theta = \angle(\vec{p}_\mu, \vec{p}_e) = \cos^{-1}(\hat{P}_e \cdot \hat{P}_\mu)$$

$$\sigma = \frac{G^2 s}{8\pi} \int_{-1}^{1} d\cos\theta\, (1 + \cos\theta)^2 = \frac{G^2 s}{3\pi} \longrightarrow \infty \text{ as } s \to \infty \,!$$

σ violates unitarity at $\sqrt{s} \geqslant 1516\,\text{GeV}$!

Optical theorem

Outgoing state ψ_f is related to the incoming state ψ_i by the S-matrix via $\psi_f \equiv S\psi_i \equiv (\delta_{fi} + i\,T_{fi})\,\psi_i$

δ_{fi}: nothing happens; $i\,T_{fi}$: scattering

Conservation of probability : $SS^{+} = 1$, i.e. S is unitarity.

$$SS^{+} = (1 + iT)(1 - iT^{+}) = 1 + i(T - T^{+}) + TT^{+} = 1$$

$$\Rightarrow \quad -i(T - T^{+}) = 2\,\mathrm{Im}(T) = TT^{+}$$

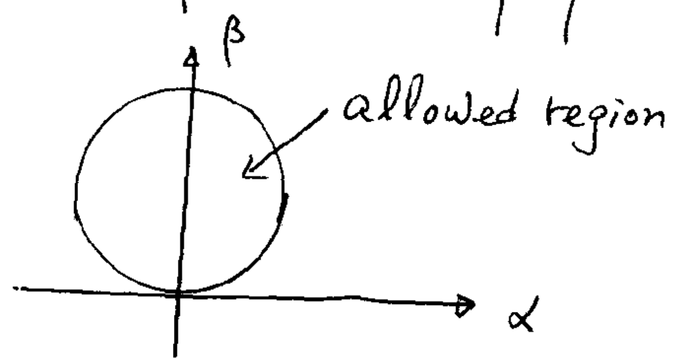
For $i = f$, same incoming & outgoing states (elastic)

$$2\,\mathrm{Im}\,T_{ii} = (TT^{+})_{ii} = \sum_k T_{ik}\,T^{+}_{ki} = \sum_k T_{ik}\,T^{*}_{ik}$$

$$\therefore \quad 2\,\mathrm{Im}\,T_{ii} = \sum_k |T_{ik}|^2 \geqslant |T_{ii}|^2$$

Let $\quad T_{ii} = \alpha + i\beta$, we have

$$2\beta \geqslant \alpha^2 + \beta^2 \quad \text{or} \quad \alpha^2 \leqslant 2\beta - \beta^2$$



$$\therefore \quad 0 \leqslant \beta \leqslant 2$$

$$\Rightarrow \beta = \mathrm{Im}\,T_{ii} = \frac{1}{2}\sum_k |T_{ik}|^2 \leqslant 2$$

$$\text{or} \quad \sum_k |T_{ik}|^2 \leqslant 4$$

P.42 $\quad \therefore$ The total crosssection $\sigma_{i\,\text{total}}$ is limited by unitarity !

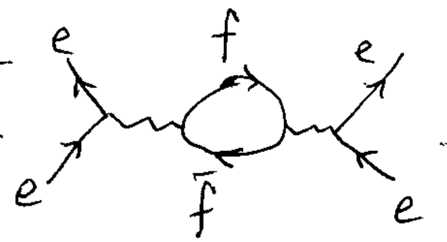
Optical theorem in diagrammatic expression

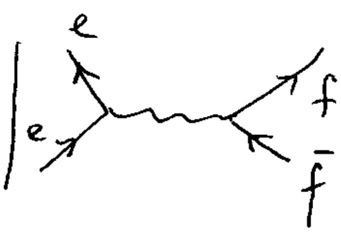
Example : photon propagator :

$$\mathrm{Im}\Big[$$  $$\Big] \sim \sum_k \Big|$$  $$\Big|^2$$

$$\left(\because \ \mathrm{Im}\, T_{ii} = \frac{1}{2} \sum_k |T_{ik}|^2 \right)$$

Attach fermion currents & specialize to one particular fermion ($f\bar{f}$) contribution :

$$\mathrm{Im}\Big[$$ 

$$\Big] \sim \Big|$$ 

$$\Big|^2$$

$$\frac{1}{s}\, \mathrm{Im}\, M^{(f)}_{i \to i} = \sigma_{tot}(e\bar{e} \to f\bar{f})$$

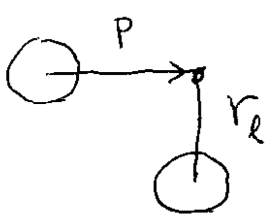
contribution from intermediate state f only with $f \neq i$

total crosssection for production of final state f with $f \neq i$

General unitarity limits

(1) Partial wave analysis leads to unitarity
for a given total angular momentum J

- scalar – scalar collision



$$\sigma_J(s) \le 4\pi\left[(r_{J+1})^2 - r_J^2\right]$$ if total black, then =

$$\sigma_J(s) \le 4\pi\left[\left(\frac{J+1}{p}\right)^2 - \left(\frac{J}{p}\right)^2\right]$$

$$\sigma_J(s) \le 4\pi\,\frac{2J+1}{s/4} = 16\pi(2J+1)/s\,.$$

- fermion – fermion (spin average $\frac{1}{2S_a+1}\,\frac{1}{2S_b+1}$)

$$\sigma_J(s) \le 4\pi(2J+1)/s$$

(2) Include all partial waves ($J=0$ to ∞)
Frossart bound

$$\sum_{J=0}^{\infty} \sigma_J(s) \le C\cdot(\ln s)^2$$

$\bar{p}p$ shows such a $(\ln s)^2$ dependence.