Lecture 4

Notations: $(\hbar = c = 1)$

- Four-vectors in space-time

$$x^\mu = (x^0, x^i) \quad i = 1, 2, 3$$
$$= (t, \vec{x}) \quad x^0 \equiv t, \ (x^i) \equiv \vec{x}$$

- Metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

$$g^{\mu\alpha} g_{\alpha\nu} = g^\mu{}_\nu = I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$\Rightarrow x_\mu = g_{\mu\nu} x^\nu = (x_0, x_i) = (t, -\vec{x})$$

- Scalar product:

$$x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu = g^{\mu\nu} x_\mu x_\nu = t^2 - \vec{x}^2$$

- Divergence

$$\begin{cases} \partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{\partial t}, \vec{\nabla}\right) & P_\mu = i\partial_\mu \\ \partial^\mu = \dfrac{\partial}{\partial x_\mu} = (\partial_t, -\vec{\nabla}) & P^\mu = i\partial^\mu \end{cases}$$

μ, ν, α (in space-time!)

$$\Box^2 = \partial^\mu \partial_\mu = \partial_t^2 - \vec{\nabla}^2$$

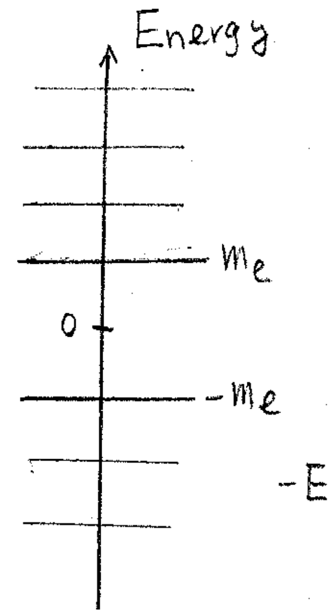
$$(\Box^2 + m^2)\phi = 0 \rightarrow -(\partial_t^2 - \nabla^2)\phi = m^2\phi$$

$$E^2 - P^2 = m^2$$

Free parti. sol. $\phi = e^{-ip_\mu x^\mu} N \rightarrow (\Box^2 + m^2)\phi = 0$

$$\phi = N e^{-i(Et - \vec{p}\cdot\vec{x})} \cdot \quad j^\mu = 2p^\mu |N|^2 \qquad E^2 = (p^2 + m^2)$$

$$\therefore E < 0 \quad \& \quad \rho = 2E|N|^2 < 0 \qquad \therefore E = \pm(\quad)^{1/2}$$



EM current $j^\mu = (\rho, \vec{j}) = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$

$$j^\mu(e^-) = -2e|N|^2(E, \vec{p})$$

$$j^\mu(e^+) = +2e|N|^2(E, \vec{p})$$

$$= -2e|N|^2(-E, -\vec{p})$$

$$e^+ \uparrow E > 0 \equiv \downarrow e^-_{(-E)<0} \qquad \uparrow t$$

$$e^{-i(+E)(+t)} = e^{-i(\cancel{E})(\cancel{t})}$$

An anti-particle going forward in time is equivalent to a particle going back in time.

Dirac Eq. linear in $P^\mu = i(\partial_t, -\nabla)$

Let $H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi \Rightarrow i\frac{\partial\psi}{\partial t} = (-i\vec{\alpha}\cdot\nabla + \beta m)\psi$ ①

requiring $H^2\psi = (\quad\quad)^2\psi = (p^2 + m^2)\psi$

$$= [\underbrace{\alpha_i^2}_{1} p^2 + \underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{0} p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0} p_i m + \underbrace{\beta^2}_{1} m^2]\psi$$

i.e. $\alpha_i^2 = \beta^2 = 1$ $\quad \{\alpha_i, \alpha_j\} = 0 \quad \{\alpha_i, \beta\} = 0$

Dirac representation: γ^0 diagonal

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma_0$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

γ_5 representation

$$\vec{\alpha} = \begin{pmatrix} -\sigma & 0 \\ 0 & \sigma \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$\beta \to$ ①

$$i\overbrace{\beta}^{\gamma^0}\frac{\partial\psi}{\partial t} = -i\overbrace{(\beta\vec{\alpha})}^{\vec{\gamma}}\cdot\nabla\psi + m\psi$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \text{②}$$

$$\gamma^\mu = (\beta, \beta\vec{\alpha})$$

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \gamma^{0\dagger} = \gamma^0, \quad \gamma^{k\dagger} = -\gamma^k$

$\gamma^{0\,2} = I \quad , \quad (\gamma^k)^2 = -I$

• Dirac Eq:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^i \partial_i = \gamma^0 \frac{\partial}{\partial t} + \vec{\gamma}\cdot\vec{\nabla}$$

• Left- and right-handed Spinors (chiral spinors):

$$\psi_L \equiv \frac{1-\gamma_5}{2}\psi \qquad \psi_R \equiv \frac{1+\gamma_5}{2}\psi \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger$$

$$\equiv P_L\,\psi \qquad \equiv P_R\,\psi \qquad P_{L,R} = (1 \mp \gamma_5)/2$$

$$P_L = P_L^\dagger, \quad P_R = P_R^\dagger, \quad P_L + P_R = 1, \quad P_L^2 = P_L, \quad P_R^2 = P_R$$

$$P_L P_R = P_R P_L = 0$$

Note 1 $\{\gamma^\mu, \gamma_5\} = 0$

2 $\begin{cases} \gamma_5 \psi_R = \psi_R \\ \gamma_5 \psi_L = -\psi_L \end{cases}$

Note 3 $\gamma_\mu = g_{\mu\nu}\gamma^\nu \begin{cases} \gamma_0 = \gamma^0 \\ \gamma_i = -\gamma^i \end{cases}$

4 $\begin{cases} \gamma^0 = \gamma^{0\dagger} \\ \vec{\gamma}^\dagger = -\vec{\gamma}^\dagger \end{cases}$

• Adjoint Dirac Eq. $\bar{\psi} = \psi^\dagger \gamma^0$ Important!

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$ How to relate $\bar{\psi}$ to ψ?

Start with Dirac Eq.

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0$$

take † $$-i\frac{\partial \psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial \psi^\dagger}{\partial x^k}(-\gamma^k) - m\psi^\dagger = 0$$ ← multiply $\psi^\dagger$ by γ^0

Use $\gamma^k\gamma^0 = -\gamma^0\gamma^k$: $$+i\frac{\partial \psi^\dagger \gamma^0}{\partial t}\gamma^0 + \frac{\partial \psi^\dagger \gamma^0}{\partial x^k}\gamma^k + m\psi^\dagger\gamma^0 = 0$$ Let $\bar{\psi} = \psi^\dagger\gamma^0$

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

Useful Representations for γ matrices

(I) γ_5 diagonal

$$\gamma^0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$

$$\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad P_R = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix} ; \; P_L = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix}$$

(II) γ^0 diagonal

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} , \quad \gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Both choices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$\{\gamma^\mu, \gamma_5\} = 0$, $\gamma^0 = \gamma^{0\dagger}$, $\vec{\gamma} = -\vec{\gamma}^\dagger$

Important identities :

- $\bar{\Psi}\gamma^\mu\psi = \bar{\Psi}\gamma^\mu(P_L + P_R)\psi \qquad \because P_L + P_R = 1$

$$= \bar{\Psi}\gamma^\mu(P_L^2 + P_R^2)\psi \qquad \because P_L^2 = P_L , \; P_R^2 = P_R$$

$$= \bar{\Psi}\gamma^\mu P_L P_L \psi + \bar{\Psi}\gamma^\mu P_R P_R \psi$$

$$= \bar{\Psi} P_R \gamma^\mu P_L \psi + \bar{\Psi} P_L \gamma^\mu P_R \psi \qquad \because \left\{ \begin{array}{l} \{\gamma^\mu, \gamma^5\} = 0 \\ \gamma^\mu P_L \equiv P_R \gamma^\mu \end{array} \right.$$

Finally $\quad \bar{\Psi} P_R = \psi^\dagger \gamma^0 P_R = \psi^\dagger P_L \gamma^0 = (P_L\psi)^\dagger \gamma^0 = \bar{\Psi}_L$

$\quad \bar{\Psi} P_L \qquad = \bar{\Psi}_R$

$$\therefore \bar{\Psi}\gamma^\mu\psi = \underbrace{\bar{\Psi}_L \gamma^\mu \Psi_L}_{\text{doublet}} + \bar{\Psi}_R \gamma^\mu \psi_R \qquad \therefore \text{Gauge Invariant}$$

P.23 $\quad \Psi_L' = e^{-i\vec{\alpha}(x)\cdot\vec{T}}\,\Psi_L \qquad$ doublet $\qquad$ singlet $\quad \psi_R' = e^{-i\beta(x)Y}\,\psi_R$

- Similarly $\bar{\psi}\gamma^\mu\gamma^5\psi = \bar{\psi}(\gamma^\mu\gamma^5)(P_L^2 + P_R^2)\psi$

$$= \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L \qquad \text{G. I.}$$

- But consider the mass term $m\bar{\psi}\psi$

$$m\bar{\psi}\psi = m\bar{\psi}P_L P_L\psi + m\bar{\psi}P_R P_R\psi$$
$$= m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$$

doublet $\quad \psi_L' = e^{-i\vec{\alpha}\cdot\vec{I}}\psi_L$

$\bar{\psi}_R' = e^{i\beta Y}\bar{\psi}_R$

singlet

$= $ arbitary values ! $\quad \therefore m = 0$

All fermions are massless !!!

- Dirac Eq in γ_5 diagnol representation.

$$(i\gamma^\mu\partial_\mu - m)\psi = (i\gamma^0\partial_t + i\vec{\gamma}\cdot\vec{\nabla})\psi = 0$$

$$\begin{pmatrix} 0 & -i\partial_t + i\vec{\sigma}\cdot\nabla \\ -i\partial_t - i\vec{\sigma}\cdot\nabla & 0 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \chi_- \end{pmatrix} - m\begin{pmatrix} \xi_+ \\ \chi_- \end{pmatrix} = 0$$

$$(-i\partial_t + i\vec{\sigma}\cdot\nabla)\chi_- - m\xi_+ = 0$$
$$(-i\partial_t - i\vec{\sigma}\cdot\nabla)\xi_+ - m\chi_- = 0$$

If $m = 0$, eqs. decouple $\quad i\partial_t\chi_- = -\vec{\sigma}\cdot(i\nabla)\chi_-$

$$i\partial_t\xi_+ = \vec{\sigma}(-i\nabla)\xi_+$$

Try $\xi_+(t,\vec{x}) = \xi_+(E,\vec{p})\, e^{-iEt + i\vec{p}\cdot\vec{x}}$

$\chi_-(t,\vec{x}) = \chi_-(E,\vec{p})$,,

$$E\chi_- = -\vec{\sigma}\cdot\vec{p}\,\chi_-$$

$$E\xi_+ = +\vec{\sigma}\cdot\vec{p}\,\xi_+$$

For particles $E = |E| \sim |\vec{p}| > 0$

$$\vec{\sigma}\cdot\hat{p}\,\chi_- = -\chi_-$$

$$\vec{\sigma}\cdot\hat{p}\,\xi_+ = +\xi_+$$

Use $\vec{S} = \vec{\sigma}/2$ to define helicity operator

$$h = \vec{S}\cdot\hat{p} = \frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$$

$h\chi_- = -\frac{1}{2}\chi_-$

$h\xi_+ = \frac{1}{2}\xi_+$

In 4-component notation

$$\vec{S} = \begin{pmatrix} \frac{\vec{\sigma}}{2} & 0 \\ 0 & \frac{\vec{\sigma}}{2} \end{pmatrix}$$

$$\vec{S}\cdot\vec{p}\,\psi = \begin{pmatrix} \frac{\vec{\sigma}\cdot\hat{p}}{2} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\hat{p}}{2} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \chi_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \chi_- \end{pmatrix}$$

- For massless fermions, helicity $(2\vec{S}\cdot\hat{p}) = \frac{1}{2}\gamma_5\psi$ is the same as chirality (γ_5)!
- For massless antiparticles with $s = 1/2$, helicity $(2\vec{S}\cdot\hat{P})$ is the opposite ~~of~~ chirality

- For massive fermions, ξ_+ and χ_- are eigen states of chirality, i.e. $\gamma_5\begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} = +\begin{pmatrix} \xi_+ \\ 0 \end{pmatrix}$ $\quad \gamma_5\begin{pmatrix} 0 \\ \chi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \chi_- \end{pmatrix}$ but they are not helicity eigenstates, due to the mass term in Dirac Eq.