Lecture II outline

Details, see Sym. & Quarks Chapter II in Q&L

2.1 Spin $|S, M_s\rangle$

$-\frac{1}{2}$ | $\frac{1}{2}$ M_s, I_3

Iso-spin $|I, \cancel{M_I}\, I_3\rangle$

$|1,1\rangle = \uparrow\uparrow$ $\quad uu \quad u\bar{d}$

$|1,0\rangle = (\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}$ $\quad (ud+du)/\sqrt{2} \quad u\bar{u} - d\bar{d}$

$|1,-1\rangle = \downarrow\downarrow$ $\quad dd \quad d\bar{u}$

(Use $J_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\;|j, m-1\rangle$)

2.2. Sym & Gauge

Prob of ψ state to be at ϕ state:

$$\psi' = U\psi \qquad |\langle \phi'|\psi'\rangle|^2 = |\langle \phi U^+ U \psi\rangle|^2$$

$\therefore U^+U = 1$ unitary

definition of group:

$$U(R_{1,2,3,\dots}) \text{ forms a gp.} \left\{ \begin{array}{ll} I \in gp & \text{unit} \\ U_j U_i = U_k \in gp & \text{complete} \\ U^{-1} U = I & \text{inverse} \\ (U_i U_j) U_k = U_i (U_j U_k) & \text{associative} \end{array} \right.$$

$$E_{\psi\phi} = \langle \phi'|H|\psi'\rangle = \langle \phi|U^+ H U|\psi\rangle = \langle \phi H \psi\rangle$$

$$\therefore \quad [U, H] = UH - HU = 0$$

Rotation w.r. axis 3

Infinitesmal $\qquad$ finite

$$U = 1 - i\varepsilon J_3 \longrightarrow e^{-i\varepsilon J_3}$$

$$1 = U^+U = (1 + i\varepsilon J_3^+)(1 - i\varepsilon J_3) = 1 + i\varepsilon(J_3^+ - J_3) + O(\varepsilon)^2$$

$\therefore J_3^+ = J_3$ hermitian

Find J_3 !

9)

$$\psi'(\vec{r}) = \psi(R^{-1}\vec{r}) = U\psi \qquad \vec{r}' = R\vec{r}$$

$$= \psi(x+\epsilon y,\ y-\epsilon x,\ z)$$

$$= \psi(\vec{r}) + \varepsilon\left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right) \qquad P_x = -i\frac{\partial}{\partial x}$$

$$= \left[1 - i\varepsilon(xP_y - yP_x)\right]\psi \qquad P_y = -i\frac{\partial}{\partial y}$$

$$= (1 - i\varepsilon J_3)\psi$$

$$\therefore J_3 = xP_y - yP_x \quad \text{or} \quad \vec{J} = \vec{r}\times\vec{P}$$

$$U(\theta) = (U(\epsilon))^n = \left(1 - i\frac{\theta}{n}J_3\right)^n \xrightarrow[n\to\infty]{} e^{-i\theta J_3}$$

$$\therefore [J_j, J_k] = i\varepsilon_{ijk} J_i$$

J^2, J_3 have e.v.

$$[J^2, J_i] = 0$$

$$J^2|j,m\rangle = j(j+1)|j,m\rangle$$

$$J_3|j,m\rangle = m|j,m\rangle$$

2.3 SU_2

Traceless

$$J = \frac{\sigma_i}{2} \;;\; \sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$U(\theta_i) = e^{-i\theta_i \sigma_i / 2} \text{ forms } SU_2 \quad \because |e^{i\sigma_i}| = e^{i\,\mathrm{Tr}[\sigma_i]} = 1$$

Basis chosen to be e. v. of J_3

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \uparrow, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \downarrow$$

$$\boxed{J_{\frac{1}{2}\pm} = \frac{\sigma_1 \pm i\sigma_2}{2}}$$

$$e^{ix} = 1 + ix + i^2\frac{x^2}{2!} + \cdots$$

2.4 Combined Representation

$$\vec{J} = \vec{J}_A + \vec{J}_B$$

$$J = |J_A - J_B|, \;\cdots\cdots\; |J_A + J_B| \;;\; M = m_A + m_B$$

$|J_A, J_B, J, M\rangle = \sum_{m_A, m_B} C \;|\, \partial_A \partial_B m_A m_B\rangle$, C obtained using step down operator on the highest M or $m_{A,B}$ state:

$$(J_A + J_B)_- : |J_A, J_B, J, M = J\rangle = |J_A, J_B, m_A = J_A, m_B = J_B\rangle$$

$J_A = J_B = \frac{1}{2} \longrightarrow J = 0, \text{ or } 1$

$$2 \otimes 2 = \overset{1}{3} \oplus \overset{0}{1}$$

$$2 \otimes (\ \ldots\ ) = (3 \otimes 2) \oplus (1 \otimes 2)$$

$$= 4 \oplus 2 \oplus 2$$

$$\underbrace{(1,\ 0)} \times \tfrac{1}{2} = \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}$$

additive

2.5. P & C $\quad \pm 1$ e.v.

multiplicative

2.6 SU_2 Isospin

$$[I_i, I_j] = i\,\varepsilon_{ijk}\, I_k$$

$$I_i = \frac{1}{2}\tau_i$$

e.g. $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ most positive Q has highest I_3.

2.7. Isospin of anti-particle $\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\,\sigma_2$.

$$\begin{pmatrix} p \\ n \end{pmatrix}' = e^{-i\pi\sigma_2/2}\begin{pmatrix} p \\ n \end{pmatrix} = -i\sigma_2 \begin{pmatrix} p \\ n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} -n \\ p \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \bar p \\ \bar n \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar p \\ \bar n \end{pmatrix}$$

$$Cp = \bar p \qquad Cn = \bar n \qquad \begin{pmatrix} \bar n \\ \bar p \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \bar n \\ \bar p \end{pmatrix}$$

11)

$$-\bar{n}' = -\bar{p} \qquad \begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix}$$
$$\bar{p}' = -\bar{n}$$

$$\begin{array}{c} (1,1) \\ (1,0) \\ (1,-1) \end{array} \begin{pmatrix} p(-\bar{n}) \\ \frac{1}{2}(p\bar{p} - n\bar{n}) \\ n\bar{p} \end{pmatrix} \qquad \frac{1}{2}(p\bar{p} + n\bar{n}) = (0,0)$$

2.8 SU_3 unitary 3×3 matrice with $|U| = 1$ traceless

Rank # diagonal matrices i.e. commuting $= 2$

color $\quad \bar{3} \otimes 3 = 8 \oplus 1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & -2 \end{pmatrix}$$

e.v. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

12)