

8.513 Problem Set 4

(Dated: October 5, 2004)

Due Oct 12

1. Calculate the temperature dependence of the specific heat of a three dimensional superfluid at low temperature. Assume that the sound wave velocity is some number c .

2. Melting curve of He-4 at low temperature

At low temperatures, application of pressure (around 25 atm) can solidify He-4. Close to the zero temperature “melting” point, it is found that heating the superfluid actually causes a transition to the solid! In other words, the melting curve is such that the pressure required to solidify He-4 is smallest at a non-zero temperature. In this problem, you will explain this observation.

- (i) First remember that a very useful guide to the shape of the melting curve is the Clausius-Clapeyron relation:

$$\left[\frac{dP}{dT} \right]_{\text{melting}} = \frac{s_l - s_s}{v_l - v_s} \quad (1)$$

where (v_l, s_l) and (v_s, s_s) are the specific volumes and specific entropies of the liquid and solid phases respectively. Prove this relation.

- (ii) The low temperature entropy is determined by the nature of the low energy excitations. In the solid, these are the usual phonons - one longitudinal and two transverse. As a simple model, assume that all three modes have the same velocity c , and calculate the low temperature entropy. In the superfluid, there is only a single phonon. Assuming that this too has the same velocity, calculate the low temperature entropy. Assume further that the density changes very little in going from the liquid to the solid phase so that $v_l \approx v_s$. Which phase has the higher entropy?

- (iii) As might be expected, the solid has a slightly higher density than the liquid. Show how the shape of the melting curve can now be understood.

3. In this problem you will examine the stability of the broken symmetry state of the superfluid to quantum and thermal fluctuations. A convenient approach is to proceed as follows. Recognizing that the dangerous fluctuations are ones with low energy, focus on the fluctuations of the phase of the order parameter.

- (i) Consider $\langle \psi \rangle \approx \psi_0 \langle e^{i\theta(x)} \rangle$. In the harmonic approximation, the action governing the fluctuations of θ is quadratic. Show that in such a case

$$\langle e^{i\theta(x)} \rangle = e^{-\frac{1}{2}\langle \theta^2(x) \rangle} \quad (2)$$

(Hint: First establish this result for simple Gaussian integrals over a single variable, then generalize appropriately).

- (ii) Now express $\langle \theta^2(x) \rangle$ in terms of the Fourier components of θ . Note that the action for θ describes an infinite number of coupled harmonic oscillators. In the Fourier basis, oscillators for different \vec{k} values decouple. Hence calculation of the required average reduces to a straightforward calculation for each such simple harmonic oscillator. Use this understanding to obtain an expression for $\langle \theta^2(x) \rangle$.

- (iii) Now consider the stability issue. Consider the order parameter reduction in various spatial dimension $d = 1, 2, 3$. Stability of the superfluid state is signalled by a finite reduction of the order parameter, *i.e.* $\langle \theta^2(x) \rangle$ is some finite number. Conversely if $\langle \theta^2(x) \rangle$ diverges, the broken symmetry state is unstable. At zero temperature, in what dimension is the superfluid stable? How does the answer change at small non-zero temperature?