

The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseWare continue to offer high quality educational resources for free. To make a donation or view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu.

PROFESSOR: OK. Let's start. So we are back to calculating partition functions for Ising models. So again some in general hypercubic lattice where you have variables σ_i being minus plus 1 at each site with a tendency to be parallel. And our task is to calculate the partition function which for n sites amounts to summing over 2^n binary configurations of a weight that favors [INAUDIBLE] to be in the same state with a strength K .

OK. So the procedure that we are going to follow is this high temperature expansion, that is writing this as hyperbolic cosine of $K \sum \sigma_i \sigma_j$. And then as we discussed, actually this \sum would have to write as a product over all bonds. And then essentially to each bond on this lattice I would have to either assign 1 or $t \sigma_i \sigma_j$. Binary choice now moves to bonds.

And then we saw that in order to make sure that following the summation over σ these factors survived we have to make sure that we construct graphs where out of each site go out an even number of bonds. And we rewrote the whole thing as $2^n \cos K$ to the number of bonds, which for a hypercubic lattice is dn . And then we had a sum over graphs where we had an even number of bonds per site. And the contribution of the graph was basically t to the power of the number of bonds.

So I'm going to call this sum over here s . After all, the interesting part of the problem is captured in this factor s , these $2^n \cos K$ to the power of dn are perfectly well-behaved analytical functions. We are looking for something that is singular. So this s I can either pick one from all of the factors-- so to the lowest order in t I would have to do to one. And then we saw that the next order correction would be something like a square, would be this t to the 4th. And then objects that are more

complicated versions.

And I can write all of those as a sum over kind of graphs that I can draw on the lattice by going around and making a loop. But then I will have graphs that will be composed of two loops, for example, that are disconnected. Since this can be translated all over the place this would have a factor of n for a lattice that is large for getting side effects and edge effects. This would have a contribution once I slide these two with respect to each other. That is, of the order of n squared. And then I would have things that would three loops and so forth.

Now based on what we have seen before it is very tempting to somehow exponentiate this sum and write it as exponential of the sum over objects that have a single loop. And I will call this new sum actually s prime for reasons to become apparent because if I start to expand this exponential, what do I get?

I will certainly start to get 1, then I will have the sum over single loop graphs plus one half whatever is in the exponent over here squared, 1 over 3 factorial whatever is in the exponent, which is this sum cubed, and so forth.

And if I were to expand this thing that is something squared, I will uncertainty get terms in it that would correspond to two of the terms in the sum. Here I would get things that would correspond to three of the terms in the sum. And the communitorial factors would certainly work out to get rid of the 2 and the 6 , depending on-- let's say I have loop a , b , or c , I could pick a from the first sum here, b from the second, c from the third or any permutation thereafter that would amount to c . I would have a factor of c abc .

But s definitely is not equal to s prime because immediately we see that once I square this term a plus b squared, in addition to a squared ab I will also get a squared and b squared, which corresponds to objects such as this. One half of the same graph repeated twice. Right? It is the a squared term that is appearing in this sum. And this clearly has no analog here in my sum s . I emphasize that in the sum s each bond can occur either zero or one time.

Whereas if I exponentiate this you can see that s prime certainly contains things that potentially are repeated twice. As I go further I could have multiple times. Also when I square this term I could potentially get 1 factor from 1, another factor, which is also a loop from the second term, the second in this product of two brackets, which once I multiply them happen to overlap and share something. OK? All right.

So very roughly and hand-wavingly, the thing is that s has loops that avoid each other, don't intersect. Whereas s prime has loops that intersect. So very naively s is sum over, if you like, a gas of non-intersecting loops. Whereas s prime is a sum over gas of what I could call phantom loops. They're kind of like ghosts. They can go through each other. OK? All right.

So that's one of a number of problems. Now in this lecture we are going to ignore this difference between s and s prime and calculate s prime, which therefore we will not be doing the right job. And this is one example of where I'm not doing the right job. And as I go through the calculation you may want to figure out also other places where I don't know correctly reproduce the sum s in order to ultimately be able to make a calculation and see what it comes to.

Now next lecture we will try to correct all of those errors, so you better keep track of all of those errors, make sure that I am doing everything right. Seems good-- all right. So let's go and take a look at this s prime.

So s prime, actually, \log of s prime is a sum over graphs that I can draw on the lattice. And since I could exponentiate that, there was no problem in the exponentiation involved over here. Essentially I have to, for calculating the log, just calculate these singly connected loops. And you can see that that will work out fine as far as extensivity is concerned because I could translate each one of these figures of the loops over the lattice and get the factors of n . So this is clearly something that is OK.

Now since I'm already making this mistake of forgetting about the intersections between loops I'm going to make another assumption that is in this sum of single loops. I already include things where the single loop is allowed to intersect itself. For

example, I'm going to allow as a single loop entity something that is like this where this particular bond will give a factor of t squared. Clearly I should not include this in the correct sum. But since I'm ignoring intersections among the different groups and I'm making it phantom, let's make it also phantom with respect to itself, allow it to intersect itself [INAUDIBLE]. It's in the same spirit of the mistake that we are going to do.

So now what do we have? We can write that \log of s prime is this sum over loops of length l and then multiply it by t to the l . So basically I say that I can draw loops, let's say of size four, loops of size six. Each one of them I have to multiply by t to the l . Of course, I have to multiply with the number of loops of length l . OK?

And this I'm going to write slightly differently. So I'm going to say that \log of s prime is sum over this length of the loop. All the loops of length l are going to contribute a factor of t to the l . And I'm going to count the loops of length l that start and end at the origin. And I'll give that a symbol [? w₀₀. ?]

Actually, very soon I will introduce a generalization of this. So let me write the definition. I define a matrix that is indexed by two side of the lattice and counts the number of walks that I can have from one to the other, from i to j in l steps. So this is number of walks of length l from i to j . OK?

So what am I doing here? Since I am looking at single loop objects, I want to sum over, let's say, all terms that contribute, in this case t to the 4. It's obvious because it's really just one shape. It's a square, but this square I could have started at anywhere on the lattice. And this slight factor, which captures the extensivity, I'll take outside because I expect this \log of s to be extensive. It should be proportional to n . So one part of it is essentially where I start to draw this loop. So I say that I always start with the loops that I have at point zero.

Then I want to come back to myself. So I indicate that the end point should also be zero. And if I want to get a term here, this is a term that is t to the fourth. I need to know how many of such blocks I have. Yes?

AUDIENCE: Are you allowing the loop to intersect itself in this case or not?

PROFESSOR: In this case, yes. I will also, when I'm calculating anything to do with s prime I allow intersection. So if you are asking whether I'm allowing something like this, the answer is yes.

AUDIENCE: OK.

PROFESSOR: Yeah.

AUDIENCE: And are we assuming an infinitely large system so that--

PROFESSOR: Yes. That's right. So that the edge effects, you don't have to worry about. Or alternatively you can imagine that you have periodic boundary conditions. And with periodic boundary conditions we can still slide it all over the place. OK? But clearly then the maximal size of these loops, et cetera, will be determined potentially by the size of the lattice.

Now this is not entirely correct because there is an over-counting. This 1 one square that I have drawn over here, I could have started from this point, to this point, this point. And essentially for something that has length l , I would have had l possible starting points. So in order to avoid the over-counting I have to divide by l . And in fact I could have started walking along this direction, or alternatively I could have gone in the clockwise direction. So there's two orientations to the walk that will take me from the origin back to the origin in l steps.

And not to do the over-counting, I have to divide by $2l$. Yes?

AUDIENCE: If we allow walking over ourselves is it always a degeneracy of $2l$?

PROFESSOR: Yes. You can go and do the calculation to convince yourself that even for something as convoluted as that, [INAUDIBLE].

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK. So this is what we want to calculate. Well, it turns out that this entity actually

shows up somewhere else also. So let me tell you why I wanted to write a more general thing. Another quantity that I can try to calculate is the spin-spin correlation. I can pick spin zero here and say spin r here, some other location. And I want to calculate what is the correlation between these two spins. OK?

So how do I have to do that for the Ising model? I have to essentially sum over all configurations with an additional factor of $\sigma_0 \sigma_r$ of this weight, $e^{-K \sum_{ij} \sigma_i \sigma_j}$, appropriately normalized, of course, by the partition function. And I can make the same transformation that I have on the first line of these exponential factors to write this as a sum over σ_i , I have $\sigma_0 \sigma_r$, and then product over all bonds of these factors of $1 + t \sigma_i \sigma_j$.

The factors of $2^n \cos K$ will cancel out between the numerator and the denominator. And basically I will get the same thing. Now of course the denominator is the partition function. It is the sum S that we are after, but we can also, and we've seen this already how to do this, express the sum in the numerator graphically. And the difference between the numerator and the denominator is that I have an additional σ sitting here, an additional σ sitting there, that if left by themselves will average out to 0.

So I need to connect them by paths that are composed of factors of $t \sigma_i \sigma_j$, originating on one and ending on the other. Right? So in the same sense that what is appearing in the denominator is a sum that involves these loops, the first term that appears in the numerator is a path that connects zero to r through some combination of these factors of t , and then I have to sum over all possible ways of doing that.

But then I could certainly have graphs that involves the same thing. And the loop, there is nothing that is against that, or the same thing, and two loops, and so forth. And you can see that as long as, and only as long as I treat these as phantom objects that can pass through each other I can factor out this term and the rest of $1 + t$, loop plus 2 loop, is exactly what I have in the denominator.

So we see that under the assumption of phantomness, if phantom then this becomes really just the sum over all paths that go from 0 to r . And of course the contribution of each path is how many factors of t I have. Right? So I have to have a sum over the length of this path l . Paths of length l will contribute a factor of t to the l . But there are potentially multiple ways to go from i_0 to r in l steps.

How many ways? That's precisely what I call this $0w$ of l . Yes?

AUDIENCE: Why does a graph that goes from 0 to r in three different ways have [INAUDIBLE]?

PROFESSOR: OK. So you want to have to go from 0 to r , you want to have a single path, and then you want that path to do something like this?

AUDIENCE: Yeah. That doesn't [INAUDIBLE].

PROFESSOR: That's fine. If I ignore the phantomness condition this is the same as this multiplied by this, which is a term that appears in the denominator and cancels out.

AUDIENCE: But you're assuming that you have the phantom condition. So this is completely normal. It doesn't matter.

PROFESSOR: I'm not sure I understand your question. You say that even without the phantom condition this graph exists.

AUDIENCE: With them phantom condition--

PROFESSOR: Yes.

AUDIENCE: --this graph is perfectly normal.

PROFESSOR: Even without the phantom condition this is an acceptable graph. Yeah. OK. Yeah?

AUDIENCE: So what does phantomness mean? Why then we can simplify only a [INAUDIBLE]?

PROFESSOR: OK. Because let's say I were to take this as a check and multiply it by the denominator. The question is, would I generate a series that is in the numerator? OK? So if I take this object that I have said is the answer, I have to multiply it by this

object. And make sure that it introduces correctly the numerator. The question is, when does it? I mean, certainly when I multiply this by this, I will get the possibility of having a graph such as this. And from here, I can have a loop such as this. And the two of them would share a bond such as that. So in the real Ising model, that is not allowed. So that's the phantomness condition that allows me to factor these things. OK? All right.

So we see that if I have this quantity that I have written in red, then I can calculate both correlation functions, as well as the free energy, log of the partition function within this phantomness assumption. So the question is, can I calculate that? And the answer is that calculating number of random walks is one of the most basic things that one does in statistical physics, and easily accomplished as follows.

Basically, I say that, OK, let's say that I start from 0, actually let's do it 0 and r , and let's say that I have looked at all possible paths that have l steps and end up over here.

So this is step one, step two, step number three. And the last one, step l , I have purposely drawn as a dotted line. Maybe I will pull this point further down to emphasize that this is the last one. This is $l - 1$ is the previous one. So I can certainly state that the number of walks from 0 to r in l steps.

Well, any walk that got to r in l steps, at the $l - 1$ step had to be somewhere. OK? So what I do is I do a number of walks from 0 to r prime, I'll call this point r prime, in $l - 1$ steps. And then times the number of ways, or number of walks from r prime to r in one step. So before I reach my destination the previous step I had to have been somewhere, I sum over all possible various places where that somewhere has to be, and then I have to make sure that I can reach from that somewhere in one step to my destination. That's all that sum is, OK?

Now I can convert that to mathematics. This quantity is start from 0, take l steps, arrive at r . By definition that's the number. And what it says is that this should be a sum over r prime, start from 0, take $l - 1$ steps, arrive at r prime. Start from r prime, take one step, arrive at your destination r , sum over r prime. OK?

Now these are n by n matrices that are labeled by l . Right? So these, being n by n matrices, this summation over r prime is clearly a matrix multiplication. So what that says is that summing over r prime tells you that that is w_1, w of l minus 1, 0. And that is true for any pair of elements, starting and final points. So basically, quite generically, we see that w , the matrix that corresponds to the count for l steps, is obtained from the matrix that corresponds to the count for one step multiplying that of l minus 1 steps. And clearly l can keep going. w_{l-1} , I can write as w_1, w of l minus 2, and so forth.

And ultimately the answer is none other than the entity that corresponds to one step raised to the l power. And just to make things easier on my writing, I will indicate this as t to the l where t stands for this matrix for one set. OK?

This condition over here that I said in words that allows me to write this in this nice matrix form is called the Markovian condition. The kind of walks that I have been telling you our Markovian in the sense that they only they depend on where you came from at the last step. They don't have memory of where you had been before.

And that's what enables us to do this. And that's why I had to do the phantom condition because if I really wanted to say that something like this has to be excluded, then the walk must keep memory of every place that it had been before. Right? And then it would be non-Markovian. Then I wouldn't have been able to do this nice calculation. That's why I had to do this phantomness assumption so that I forgot the memory of where my walk was previously. OK?

So this matrix, where you can go in one step, is really the matrix of who is connected to whom. Right? So this tells you the connectivity. So for example, if I'm dealing with a 2D square lattice the sides on my lattice are labeled by x and y . And I can ask, where can I go in one step if I start from x and y ? And the answer is that either x stays the same and y shifts by one, or y stays the same and x shifts by one. These are the four non-zero elements of the matrix that allows you to go on the square lattice either up, down, right, or left. And the corresponding things that you would have for the cube or whatever lattice. OK?

Now you look at this and you can see that what I have imposed clearly is such that for a lattice where every side looks like every equivalent side this is really a function only of the separation between the two points. It's an n by n matrix. It has n squared elements, but the elements really are essentially one column that gets shifted as you go further and further down in a very specific way.

And whenever you have a matrix such as this translational symmetry implies that you can diagonalize it by Fourier transformation. And what do I mean by that? This is, I can define a vector q such that it's various components are things like e to the $i q \cdot r$ in whatever dimension. Let's normalize it by square root of n . And I should be sure that that is an eigenvector.

So basically my statement is that if I take the matrix t , act on q , then I should get some eigenvalue in the vector back. And let's check that for the case of the 2D system. So for the 2D system, if I say that $x' y' t q_x q_y$, what is it? Well, that is $x' y' t x$ prime, y prime, the entity that I have calculated here, x prime, y prime q_x, q_y . And of course, I have a sum over x prime and y prime. That's the matrix product.

And so again, remember this entity is simply e to the $i q_x x$ prime plus $q_y y$ prime, divided by square root of n . And because this is a set of delta functions, what does it do? It basically sets x prime either to x plus 1, or x minus 1, y prime either to y or y minus 1, y minus 1. You can see that you always get back your e to the $i q_x x$ plus $q_y y$ with a factor of root n .

So essentially that by the delta functions just changes the x primes to x at the cost of the different shifts that you have to do over there, which means that you will get a factor of e to the $i q_x$, e to the minus $i q_x$, with q_y not changing, or e to the $i q_y$, and e to the minus $i q_y$ with the x component not changing. So this is the standard thing that you have seen, is none other than $2 \cos$ of q_x , plus \cos of q_y . And so we can see that quite generally in the d -dimensional hypercube, my t of q is going to be the sum over all d components of these factors of \cos of $q \cdot \alpha$. And that's about it, OK?

So why did I bother to do this diagonalization? The answer is that that now allows me to calculate everything that I want. So, for example, I know that this quantity that I'm interested in, $\sum_r \sum_l$, is going to be a sum over l , t to the l , then 0 . w_l is t to the l times r . Right?

Now this small t , I can take inside here and do it like this. And if I want, I can write this as the 0 r component of a sum over l t raised to the power of l . So it's a new matrix, which is essentially sum over l , small t times this connectivity matrix to the l -th power. This is a geometric series. We can immediately do the geometric series. The Answer is $0, 1$ over 1 minus tt going all the way to r . OK? And the reason I did this diagonalization is so that I can calculate this matrix element, because I don't really want to invert a whole matrix, but I can certainly invert the matrix when it is in the diagonal basis because all I have to do is to invert pure numbers.

So what is done is to go to the Fourier basis and rotate to the Fourier basis calculate this. It is diagonal in this basis. I have q r . And so what is that? These are these exponentials here evaluated at the origin, so it's just 1 over root n . This is 1 over root n . This is e to the i q dot r over root n . This is just the eigenvalue that I have calculated over here. So this entity is none other than a sum over q , e to the i q dot r divided by n , two factors of square root of n . 1 over 1 minus t -- well, actually let's write it $2t$ -- sum over α cosine of q α .

And then of course I'm interested in big systems. I replace the sum over q with an integral over q , 2π to the d density of states. In going from there to there, there's a factor of volume, the way that I have set the unit of length in my system. The volume is actually the number of particles that I have. So that factor of n disappears. And all I need to do is evaluate this factor of 1 minus $2t$ sum over α cosine of q α integrated over q Fourier transformed. OK? Yes?

AUDIENCE: So I notice that you have a sum over q , but then you also have a sum α q α .

PROFESSOR: Right.

AUDIENCE: Is there a relationship between the q and the q alpha or not?

PROFESSOR: OK. So that goes back here. So when I had two dimensions, I had q_x and q_y . Right? And so I labeled, rather than x and y , with q_1 and q_2 . So the index alpha is just a number of spatial dimensions. If you like, this is also dq_1 , dq_2 , dq_d .

AUDIENCE: OK. [INAUDIBLE].

PROFESSOR: OK. All right. So we are down here. Let's proceed. So what is going to happen? So suppose I'm picking two sides, 0 and r , let's say both along the x direction, some particular distance apart. Let's say seven, eight apart. So in order to evaluate this I would have an integral if this is the x direction of something like e to the $i q_x$ times r . Now when I integrate over q_x the integral of e to the $i q_x$ times r would go to 0 .

The only way that it won't go to 0 is from the expansion of what is in the denominator. I should bring on enough factors of e to the minus $i q_x$, which certainly exist in these cosine factors, to get rid of that. So essentially, the mathematical procedure that is over here is to bring in sufficient factors of e to the minus $i q \cdot r$ to eliminate that. And the number of ways that you are going to do that is precisely another way of capturing this entity, which means that clearly if I'm looking at something like this, and I mean the limit that t goes to be very, very small so that the lowest order in t contributes, the lowest order t would be the shortest path that joins these two points.

So it is like connecting these two points with a string that is very tight. So that what I am saying is that the limit as t goes to 0 of something like σ_0 , σ_r is going to be identical to t to the minimum distance between 0 and r . Actually, I should say proportional, is in fact more correct. Because there could be multiple shortest paths that go between two points. OK?

Now let's make sense. There's kind of an exponential. Essentially I start with the high temperature limit where the two things don't know anything about each other. So σ_0 , σ_r is going to be 0 . So anything that is beyond 0 has to come from somewhere in which the information about the state of the site was conveyed all the

way over here. And it is done through passing one bond at a time. And in some sense, the fidelity of each one of those transformations is proportional to t .

Now as t becomes larger you are going to be willing to pay the costs of paths that go from 0 to r in a slightly more disordered way. So your string that was tight becomes kind of loose and floppy. And why does that become the case? Because now although these paths are longer and carry more factors of this t that is small, there are just so many of them that the entropy, the number of these paths starts to dominate. OK?

So very roughly you can state the competition between them as follows, that the contribution of the path of length l , decays like t to the l , but the number of paths roughly grows like $2d$ to the power of l . And since we have this phantomness character, if I am sitting at a particular site, I can go up, down, right, left. So at each step I have in two dimensions a choice of four, in three dimensions a choice of six, in d -dimensions a choice of $2d$.

So you can see that this is exponentially small, this is exponentially large. So they kind of balance each other. And the balance is something like e to the minus l over some typical l that we'll contribute. And clearly the typical l is going to be finite as long as $2dt$ is less than 1. So you can see that something strange has to happen at the value where t_c is such that $2dt_c$ is equal to 1.

At that point the cost of making your paths longer is more than made up by increasing the number of paths that you can have, the entropy starts to be. And you can see that precisely that condition tells me whether or not this integral exists, right? Because one point of this integral where the integrand is largest is when q goes to 0. And you can see that as q goes to 0, the value in the denominator is $1 - 2dt$. So there is precisely a pole when this condition takes place.

And if I'm interested in seeing what is happening when I'm in the vicinity of that transition, right before these paths become very large, what I can do is I can start exploring what is happening in the vicinity of that pole. So $1 - 2d$, sum over α cosine of $q \alpha$, how does it behave?

Each cosine I can start expanding around its q going to 0 as $1 - q^2$ over 2, so you can see that this is going to be $1 - 2td$. And then I would have plus $t q^2$, because I will have $q_1^2 + q_2^2 + q_d$. So this q^2 is the sum of all the q 's.

And then I do have higher order terms. Order of q to fourth, and so forth. OK? So if I'm looking in the vicinity of t going to t_c , this is roughly $t_c q^2$. This is something that goes to 0 and once I factor out the t_c , I can define a length squared in this fashion, inverse length squared. OK?