Quantum Field Theory II (8.324) Fall 2010
Assignment 6

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder chapters 10, 12, 13.
- Weinberg vol 1 chapter 12 and Vol 2 chapter 18.

Problem Set 5

1. **Furry's theorem (20 points)**

   (a) Using charge conjugation invariance to prove that the vacuum expectation value of the time-ordered product of any odd number of electromagnetic fields and/or currents vanish.

   (b) Verify directly that the one-loop contribution to the photon one-point and three-point functions vanish.

2. **A massive vector field theory (30 points)**

   Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - i\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi \tag{1}$$

   (a) Find the momentum space propagator $D_{\mu\nu}(k)$ for the massive vector field A_μ.

   (b) Show that in the large momentum limit, $D_{\mu\nu}(k)$ scales as $O(1)$ (i.e. independent of k). Convince yourself that in terms of momentum counting, this implies that A_μ has an effective dimension 2.

   (c) By explicitly counting the power of momentum inside the integrand of the amplitude, derive an explicit expression for the superficial divergence D for a diagram with E_e external ψ-lines, E_A external A_μ-lines and V vertices.

   (d) Show that the expression obtained in (c) is the same as that derived in lecture by using dimensional analysis with the effective dimension of A_μ now given by 2.

(e) Is this theory renormalizable? Why?

3. **Operator product expansions (20 points)**

Consider a free scalar field theory in *Euclidean signature* with a Lagrangian

$$\mathcal{L} = \frac{1}{2} \int d^4x \, \left((\partial\phi)^2 + m^2\phi^2 \right) \tag{2}$$

(a) Express the operator $\phi^4(x)$ in terms of normal-ordered operators.

(b) Denote $\mathcal{O}(x) =: \phi^4(x) :$ where :: stands for normal ordering. Consider the operator product expansion (as $|x| \to 0$)

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_n C_n(x)\, \mathcal{O}_n(0) \tag{3}$$

where $\mathcal{O}_n$ denotes a complete set of normal-ordered local operators built from ϕ and its derivatives. Work out the coefficients $C_n(x)$ for those $\mathcal{O}_n$ who canonical dimension $\Delta_n \leq 4$.

4. **Renormalization group properties (30 points)**

(a) Consider a coupling constant λ and a redefined coupling constant $\bar{\lambda}(\lambda)$. Find the general transformation law for the beta function, namely the relation between $\beta(\lambda)$ and $\overline{\beta}(\overline{\lambda})$. If we think of λ as a coordinate we see that β transforms as a tensor. What kind of tensor ?

(b) Assume that

$$\beta(\lambda) = b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + \cdots$$

and consider the perturbatively defined and invertible coupling constant redefinition:

$$\bar{\lambda}(\lambda) = \lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots .$$

Calculate $\bar{\beta}(\bar{\lambda})$ writing it in the form

$$\bar{\beta}(\bar{\lambda}) = \bar{b}_2\bar{\lambda}^2 + \bar{b}_3\bar{\lambda}^3 + \bar{b}_4\bar{\lambda}^4 + \cdots$$

Verify that:

i. $\overline{b}_2 = b_2$ and $\overline{b}_3 = b_3$.

ii. It is possible to make $\bar{b}_4$ anything you want by such a coupling redefinition.

iii. Let $\lambda = \lambda_F$ denote a fixed point. Show that $\bar{\lambda} = \bar{\lambda}_F$ is also a fixed point. How are the derivatives β' and $\bar{\beta}'$ related at the fixed point?

(c) Consider the differential equation for a massless coupling g

$$\mu \frac{dg}{d\mu} = -bg^2 - cg^3 - dg^4 - \cdots \tag{4}$$

with b, c, d numerical constants. Show that one can write a solution to the above equation in the form

$$\ln \frac{\mu}{\Lambda} = \frac{1}{bg(\mu)} + \frac{c}{b^2} \ln bg(\mu) + \mathcal{O}(g(\mu)) \tag{5}$$

where Λ is an integration constant, which can be considered as a dynamically generated scale. Argue that Λ is renormalization group invariant.

(d) More generally, show that in a renormalizable theory with a dimensionless coupling constant $g(\mu)$ and no other dimensional parameter (like in a non-Abelian gauge theory), dimensional transmutation happens. That is, show that $g(\mu)$ can be written in a form

$$g(\mu) = f\left(\log\left(\frac{\mu}{\Lambda}\right)\right) \tag{6}$$

with Λ a universal scale and f a function depending on the specific theory. (Inverting (5) gives a specific example of f.)

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