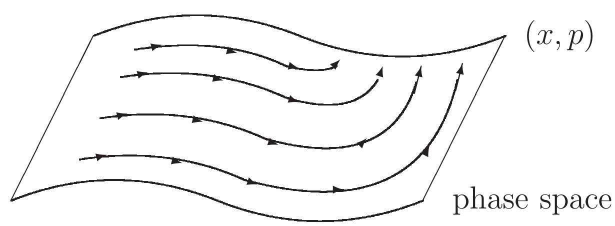
1. Fundamental Concepts

1.1 Introduction

Theoretical framework of classical physics:

- $*$ state (at fixed t) is defined by a point $\{x^i, p_i\}$ in phase space
- $*$ Poisson bracket $\{x^i, p_j\} = \delta_{ij}$ $\{FG\} = \sum_i \left(\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial G}{\partial x^i}\right)$ (flat space, symplectic mfld)
- $*$ Observables are functions $\mathcal{O}(x^i, p_j)$ on phase space [ex: $x^2 + y^2 + z^2$, $p^2/2m, \ldots$]
- $*$ Hamiltonian $H(x, p)$ defines dynamics

$$\dot{q} = \{q, H\} \text{ for } q = x^i, p_j, \ldots$$

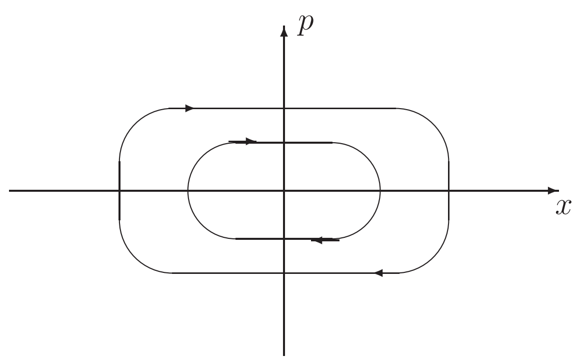


Framework describes all of mechanics. E & M including fluids, materials, etc. ...

- Simple, intuitive conceptual framework
- Deterministic
- Time-reversible dynamics

Example: Classical simple harmonic oscillator (SHO)

$$H = \frac{k}{2}x^2 + \frac{1}{2m}p^2$$



$$
\begin{aligned}
\dot{x} &= \{x, H\} = \frac{1}{m} p \\
\dot{p} &= \{p, H\} = -kx \\
& \qquad\qquad\quad (= m\ddot{x})
\end{aligned}
$$

Theoretical framework of quantum physics

* state defined by (complex) vector $|v\rangle$ in vector space $\mathcal{H}$. (Hilbert space)
* Observables are Hermitian operators $\mathcal{O}^{dagger} = \mathcal{O}$
* Dynamics: $|v(t)\rangle = e^{-iHt/\hbar}|v(0)\rangle$ (H hermitian, $\hbar$ constant)
* "Collapse postulate"

  [(subtleties: degeneracy, cts. spectrum)]

  (simple version)

  If $|\phi\rangle = \sum \alpha_i |\lambda_i\rangle$,
  with $A|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$, $\lambda_i \neq \lambda_j$
  Then with prob. $|\alpha_i|^2$, measure $A = \lambda_i$,
  $|\phi\rangle = |\lambda_i\rangle$ after measurement

This framework [(with suitable generalizations, i.e. field theory)] describes all quantum systems, and all experiments not involving gravitational forces.

– Atomic spectra (quantization of energy)

– Semiconductors transistors, (quantum tunneling)

• Counterintuitive conceptual framework

• Nondeterministic

• Irreversible dynamics

Both classical & quantum conceptual frameworks useful in certain regimes.

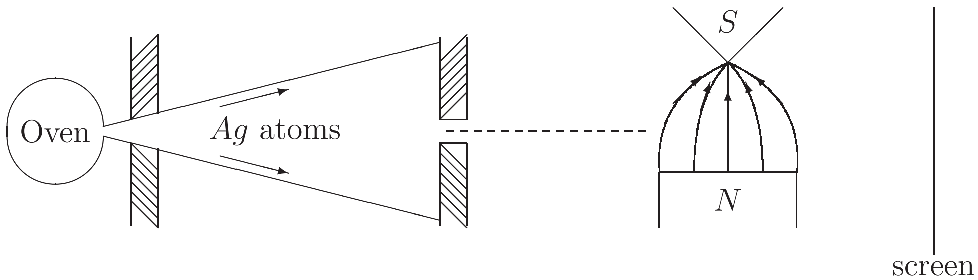
Classical picture not fundamental — replaced by QM.

Is QM fundamental?

Perhaps not, but describes all physics of current relevance to technology & society.

Simple example of QM: 2-state system (spin $\frac{1}{2}$ particle)

Stern & Gerlach 1922

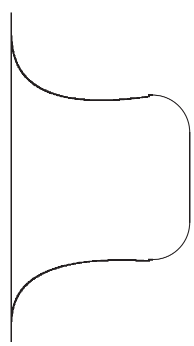


Silver: 47 electrons, angular momentum $\Longrightarrow$ μ (mag. dipole moment) from spin of 47$^{\text{th}}$ electron.

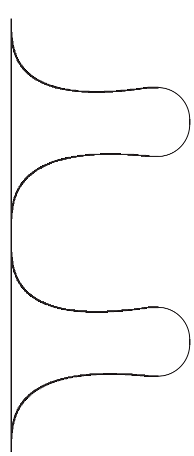
$$\begin{aligned} \text{Energy} \qquad U &= -\mu \cdot B \\ F_z &= -\frac{\partial U}{\partial Z} = \mu_z \frac{\partial B_z}{\partial Z} \end{aligned}$$

Gives force along $\hat{z}$ depending on μ_z.

Classically, expect



Actually see

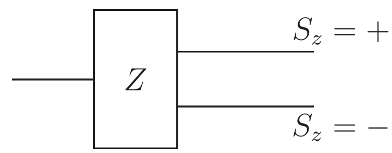


$$
\begin{aligned}
\mu_z &\approx \pm\frac{e\hbar}{2mc} \approx \frac{e}{mc} S_z \\
S_z &= \pm\frac{\hbar}{2} \qquad [\hbar = 1.0546 \times 10^{-27} \text{ ergs}]
\end{aligned}
$$

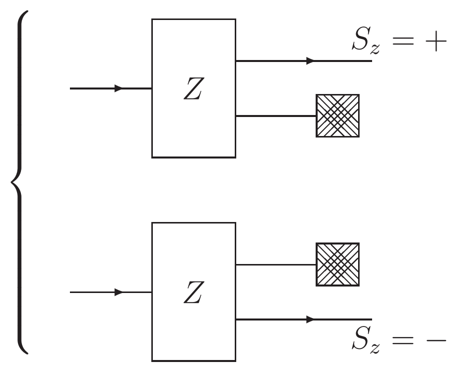
So measuring $S_z \Longrightarrow$ discrete values (2 states)

Can build sequential $S - G$ experiments, using components

splitter



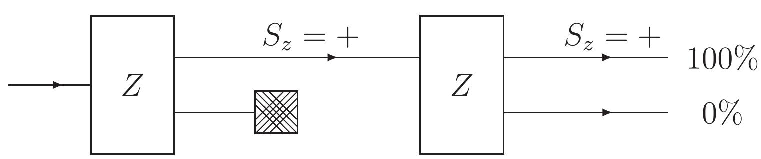
filters



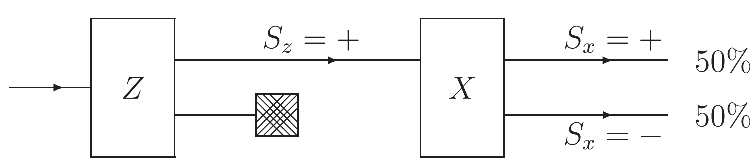
(Can also form splitter, filters on x-axis, etc.)

Single particle experiments

i)



ii)



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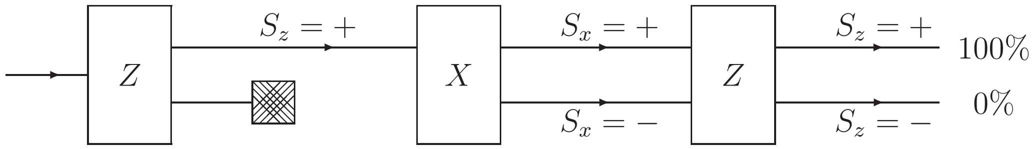
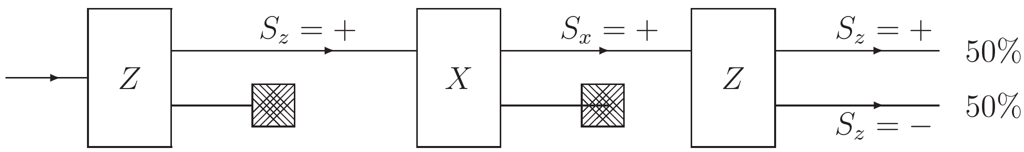
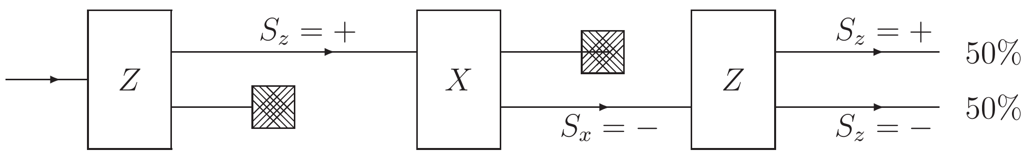


BUT

iv)



v)



Cannot simultaneously measure S_z, S_x

"incompatible observables" $\qquad S_z S_x \neq S_x S_z$

Analogous to 2-slit experiment for photons.

In iii), not measuring S_x.
in iv), v) measuring S_x.

iii) needs "interference" of probability wave — no classical interpretation.

iii), iv) $\Longrightarrow$ Irreversible, nondeterministic dynamics (assuming locality)

1.2 Mathematical Preliminaries

1.2.1 Hilbert spaces

First postulate of QM:

∗ The state of a QM system at time t is given by a vector (ray) $|\alpha\rangle$ in a complex Hilbert space $\mathcal{H}$. [[will state more precisely soon.]]

Vector spaces

A *vector space* V is a collection of objects ("vectors") $|\alpha\rangle$ having the following properties:

A1: $|\alpha\rangle + |\beta\rangle$ gives a unique vector $|\gamma\rangle$ in V.

A2: (commutativity) $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

A3: (associativity) $(|\alpha\rangle + |\beta\rangle) + |\delta\rangle = |\alpha\rangle + (|\beta\rangle + |\delta\rangle)$

A4: $\exists$ vector $|\phi\rangle$ such that $|\phi\rangle + |\alpha\rangle = |\alpha\rangle\ \forall|\alpha\rangle$

A5: For all $|\alpha\rangle$ in V, $-|\alpha\rangle$ is also in V so that $|\alpha\rangle + (-|\alpha\rangle) = |\phi\rangle$.

[(A1–A5): V is a commutative group under $+$]

For some Field F (i.e., $\mathbb{R}$, $\mathbb{C}$, with $+$, $*$ defined) *scalar multiplication* of any $c \in F$ with any $|\alpha\rangle \in V$ gives a vector $c|\alpha\rangle \in V$. Scalar multiplication has the following properties

M1: $c(d|\alpha\rangle) = (cd)|\alpha\rangle$

M2: $1|\alpha\rangle = |\alpha\rangle$

M3: $c(|\alpha\rangle + |\beta\rangle) = c|\alpha\rangle + c|\beta\rangle$

M4: $(c+d)|\alpha\rangle = c|\alpha\rangle + d|\alpha\rangle$.
V is called a "vector space over F."
$F = \mathbb{R}$: "real v.s."
$F = \mathbb{C}$: "complex v.s."

Examples of vector spaces

a) Euclidean D-dimensional space is a real v.s. $\begin{pmatrix} a_1 \\ \vdots \\ a_D \end{pmatrix}$

b) State space of spin $\frac{1}{2}$ particle is 2D cpx v.s.
states: $c_+|+\rangle + c_-|-\rangle$, $c\pm \in \mathbb{C}$ [[note: certain state are physically equivalent]]

c) space of functions $f' : [0, L] \to \mathbb{C}$

Henceforth we always take $F = \mathbb{C}$

Subspaces

$V \subset W$ is a *subspace* of a v.s. W if V satifies all props of a v.s. and is a subset of W.

[suffices for V to be closed under $+$, scalar mult.]

Ray

A ray in V is a 1D subspace $\{c|\alpha\rangle\}$.

Linear independence & bases

$|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_n\rangle$ are *linearly independent* iff

$$c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + \cdots + c_n|\alpha_n\rangle = 0$$

has only $c_1 = c_2 = \cdots = c_n = 0$ as a solution.

If $|\alpha_1\rangle, \dots, |\alpha_n\rangle$ in V are linearly independent, but all sets of $n+1$ vectors are linearly dependent, then

- V is *n-dimensional* (n may be finite, countably ∞, or uncountably ∞.)
- $|\alpha_1\rangle, \dots, |\alpha_n\rangle$ form a *basis* for the space V.

If $|\alpha_1\rangle, \dots, |\alpha_n\rangle$ form a basis for V, then any vector $|\beta\rangle$ can be expanded in the basis as

$$|\beta\rangle = \sum_i c_i|\alpha_i\rangle\,. \tag{Thm.}$$

Unitary spaces

A complex vector space V is a *unitary space* (a.k.a. inner product space)

> if given $|\alpha\rangle, |\beta\rangle \in V$ there is an inner product $\langle\alpha|\beta\rangle \in \mathbb{C}$ with the following properties:
>
> I1: $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$
> I2: $\langle\alpha|(|\beta\rangle + |\beta'\rangle) = \langle\alpha|\beta\rangle + \langle\alpha|\beta'\rangle$
> I3: $\langle\alpha|(c|\beta\rangle) = c\langle\alpha|\beta\rangle$
> I4: $\langle\alpha|\alpha\rangle \geq 0$
> I5: $\langle\alpha|\alpha\rangle = 0$ iff $|\alpha\rangle = |0\rangle$

$\langle\alpha|\beta\rangle$ is a *sesquilinear* form (linear in β), conj. lin. in $|\alpha\rangle$

Examples:

a) $V = \mathbb{C}^N$, N-tuples $|z\rangle = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}$, $z_i \in \mathbb{C}$.
$\langle z|w\rangle = z_I^* w_1 + z_2^* w_2 + \cdots + z_N^* w_N$

b) $V = \{f : [0, L] \to \mathbb{C}\}$
$\langle f|g\rangle = \int_0^L f^*(x) g(x)\, dx$

Terminology:

> $\sqrt{\langle\alpha|\beta\rangle} =$ *norm* of $|\alpha\rangle$ (sometimes, $\||\alpha\rangle\|$
> if $\langle\alpha|\beta\rangle = 0$, $|\alpha\rangle$, $|\beta\rangle$ are *orthogonal*

Dual Spaces

For a (complex) vector space V, the *dual space* V^* is the set of linear functions $\langle\beta| : V \to \mathbb{C}$. Given the inner product $\langle\beta|\alpha\rangle$, can construct isomorphism.

$$\begin{array}{rcl} |\alpha\rangle & \longleftrightarrow & \langle\alpha| \\ V & \longleftrightarrow & V^* \\ \text{note: } c|\alpha\rangle & \longleftrightarrow & c^*\langle\alpha|\,. \end{array}$$

Physics notation (Dirac "bra-ket" notation)

$$\begin{array}{rcll} |\alpha\rangle & \in & V & \text{ket} \\ \langle\beta| & \in & V^* & \text{bra} \end{array}$$

Hilbert space

A space V is *complete* if every Cauchy sequence $\{|\alpha_n\rangle\}$ converges in V
$\forall\epsilon\exists N : \||\alpha_n\rangle - |\alpha_m\rangle\| < \epsilon\ \forall m, n > N$ (i.e., $\exists|\alpha\rangle : \lim_{n\to\infty} \||\alpha\rangle - |\alpha_n\rangle\| = 0$.)

A complete unitary space is a (complex) *Hilbert space*.

Note: an example of an incomplete unitary space is the space of vectors with a finite number of nonzero entries. The sequence

$$\begin{array}{l} \{ \quad (1,\ 0,\ 0,\ \ldots\ ), \\ \quad\ (1,\ \tfrac{1}{2},\ 0,\ 0,\ \ldots\ ), \\ \quad\ (1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ 0,\ \ldots\ ) \\ \qquad\qquad \vdots \\ \qquad\qquad\qquad \} \end{array}$$

is Cauchy, but doesn't converge in V. This is *not* a Hilbert space.

A Hilbert space can be:

a) Finite dimensional (basis $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$)
   Ex. spin $\frac{1}{2}$ particle in Stern-Gerlach expt.

b) Countably infinite dimensional (basis $|\alpha_1\rangle, |\alpha_2\rangle, \ldots$)
   Ex. Quantum SHO

c) Uncountably infinite dimensional (basis $|\alpha_x\rangle$, $x \in \mathbb{R}$)

[Technical aside:

A space V is *separable* if $\exists$ countable set $D \subset V$ so that $\bar{D} = V$.
(D dense in V: $\forall\epsilon$, $|x\rangle \in V \exists |y\rangle \in D : \||x\rangle - |y\rangle\| < \epsilon$)
(a), (b) are separable, (c) is not.

non-separable Hilbert spaces are very dicey mathematically.

Generally, separability implicit in discussion — e.g., label basis $|\alpha_i\rangle$, i takes discrete values]

Orthonormal basis

An orthonormal basis is a basis $|\varphi_i\rangle$ with $\langle\varphi_i|\varphi_j\rangle = \delta_{ij}$

Any basis $|\alpha_1\rangle$, $|\alpha_2\rangle$, ... can be made orthonormal by *Schmidt orthonormalization*

$$\begin{aligned}
|\phi_1\rangle &= \frac{|\alpha_1\rangle}{\sqrt{\langle\alpha_1|\alpha_1\rangle}} \\
|\alpha_2'\rangle &= |\alpha_2\rangle - |\phi_1\rangle\langle\phi_1|\alpha_2\rangle \\
|\phi_2\rangle &= \frac{|\alpha_2'\rangle}{\sqrt{\langle\alpha_2'|\alpha_2'\rangle}} \\
|\alpha_3'\rangle &= |\alpha_3\rangle - |\phi_1\rangle\langle\phi_1|\alpha_3\rangle - |\phi_2\rangle\langle\phi_2|\alpha_3\rangle \\
&\vdots
\end{aligned}$$

Gives $|\phi_i\rangle$ with $\langle\phi_i|\phi_j\rangle = \delta_{ij}$.

If $\{|\phi_i\rangle\}$ are an orthonormal basis then for all $|\alpha\rangle$,

$$|\alpha\rangle = \sum_i c_i|\phi_i\rangle\,, \qquad c_i = \langle\phi_i|\alpha\rangle\,.$$

Can write as $|\alpha\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\alpha\rangle$. (Completeness relation)

Schwartz inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \qquad \forall|\alpha\rangle, |\beta\rangle\,.$$

Proof: write $|\gamma\rangle = |\alpha\rangle + \lambda|\beta\rangle$

$$\begin{aligned}
\langle\gamma|\gamma\rangle &= \langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^x\langle\beta|\alpha\rangle + |\lambda|^2\langle\beta|\beta\rangle \\
\text{set } \lambda &= \frac{-\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} \\
\rightarrow \qquad \langle\gamma|\gamma\rangle &= \langle\alpha|\alpha\rangle - \frac{|\langle\beta|\alpha\rangle|}{\langle\beta|\beta\rangle} \geq 0
\end{aligned}$$

$\square$

[Second postulate of QM:

* Observables are (Hermitian) operators on $\mathcal{H}$ [self-adjoint]

1.2.2 Operators

Linear operators

A linear operator from a VS V to a VS W is a transformation such that

$$A|\alpha\rangle + |\beta\rangle = A|\alpha\rangle + A|\beta\rangle \qquad \forall|\alpha\rangle,\, |\beta\rangle\,.$$

We write $A = B$ iff $A|\alpha\rangle = B|\alpha\rangle\ \forall|\alpha\rangle$.
A acts on V^* through $(\langle\beta|\ \forall A)|\alpha\rangle = \langle\beta|(A|\alpha\rangle)$ (acts on right on bras.)

Outer product

A simple class of operators are outer products $|\beta\rangle\langle\alpha|$

$$(|\beta\rangle\langle\alpha|)|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle$$

Adjoint

Recall correspondence $\begin{array}{ccc} V & \leftrightarrow & V^* \\ |\alpha\rangle & \leftrightarrow & \langle\alpha| \end{array}$

Given an operator A, define $A^\dagger$ (adjoint of A) (Hermitian conjugate) by $A^\dagger|\alpha\rangle \leftrightarrow \langle\alpha|A$

Example $(|\beta\rangle\langle\alpha|)^\dagger = \langle\alpha|\langle\beta|$. Follows that $\langle\alpha|A^\dagger|\beta\rangle = (\langle\beta|A|\alpha\rangle)^*$.

Hermitian operators

A is *Hermitian* if $A = A^\dagger$ ($\sim$ self-adjoint)

[Technical aside: mathematically, Hermitian called "symmetric". Self-adjoint iff symmetric $+A$ & $A^\dagger$ have same *domain of definition*, relevant to i.e., Dirac op. in monopole background (symmetric op. with several self-adjoint extensions) more: Reed & Simon, Jackiw

Example of domain of def: Consider $\mathcal{H} = \mathcal{L}^2(\mathbb{R}) = \{\text{funs } f : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{0} f^* f < 0\ e^{-x^2} \in \mathcal{H}$,
$\mathcal{O} =$ mult by e^{x^2}
$\mathcal{O}e^{-x^2} \notin \mathcal{H}$, so e^{-x^2} not in domain $D(\mathcal{O})$.]

Linear operators A form a vector space under addition (+ commutative, associative)

$$(A + B)|\alpha\rangle = A|\alpha\rangle + B|\alpha\rangle$$

Mult. defined by

$$(AB)|\alpha\rangle = A(B|\alpha\rangle)$$

Generally $AB \neq BA$
But $(AB)C = A(BC)$
Note: $(XY)^+ = Y^+X^+$

Identity operator: $\mathbb{1}$ $\quad\mathbb{1}|\alpha\rangle = \langle\alpha|$.
Functions of one operator $f(A) = \sum C_n A^n$ can be expanded as power series (must be careful outside ROC — can do in general if diagonalizable)

Diagonalizable operators: can always compute $f(A)$ if f defined for diagonal elements (evalues)

Functions $f(A, B)$ must have definable ordering prescription. (e.g. $e^A B e^{-A} = B + AB - BA + \cdots$)

Inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = \mathbb{1}$
Does not always exist. (Ex. if A has an ev. = 0.) Note: $BA = \mathbb{1}$ does not *imply* $AB = \mathbb{1}$ (Ex. later)

Isometries

U is an *isometry* if $U^+U = \mathbb{1}$, since preserves inner product $(\langle\beta|U^+)(U|\alpha\rangle) = \langle\beta|\alpha\rangle$

Unitary operators

U is unitary if $U^\dagger = U^{-1}$.

Example: non-unitary isometries. (Hilbert Hotel)
Consider the *shift* operator $S|n\rangle = |n + 1\rangle$ acting on $\mathcal{H}$ with countable on basis $\{|n\rangle, = 0, 1, \ldots\}$
$S = |n+1\rangle\langle n|$ satisfies $S^+S = \mathbb{1}$ but *not* $SS^+ = \mathbb{1}$. ($SS^+ = \mathbb{1} - |0\rangle\langle 0|$).
S has no (2-sided) inverse.

Projection operators

A is a *projection* if $A^2 = A$.

Ex. $A = |\alpha\rangle\langle\alpha|$ for $\langle\alpha|\alpha\rangle = 1$.

Eigenstates & Eigenvalues

If $A|\alpha\rangle = a|\alpha\rangle$ then $|\alpha\rangle$ is an *eigenstate* (eigenket) of A and a is the associated *eigenvalue.*

Spectrum

The spectrum of an operator A is its set of eigenvalues $\{a\}$
[technical aside: this is the “point spectrum”, mathematically, spectrum of A = set of $\lambda : A - \lambda\mathbb{1}$ is not invertible]

Important theorem

$\boxed{\text{If } A = A^\dagger\text{, then all eigenvalues } a_i \text{ of } A \text{ are real, and all eigenstates associated with distinct } a_i \text{ are orthogonal.}}$

Proof:

$$\begin{aligned} A|a\rangle = a|a\rangle \quad &\Rightarrow \quad \langle a|A^\dagger = \langle a|a^* \\ \Rightarrow \langle b|(A - A^+)|a\rangle \quad &= \quad (a - b^*)\langle b|a\rangle = 0 \end{aligned}$$

$$\begin{array}{lrclrcl} \text{if} & a & = & b\,, & a & = & a^* \text{ is real.} \\ \text{if} & a & \neq & b\,, & \langle b|a\rangle & = & 0 \end{array}$$

$\square$

Consequence of theorem:

For any Hermitian A, can find an O.N. set of eigenvectors $|a_i\rangle$

$$A|a_i\rangle = a_i|a_i\rangle\,, \qquad \begin{array}{c}(a_i \text{ not necc. distinct} \\ \text{— can be } \textit{degenerate})\end{array}$$

[Proof: use Schmidt orthog. for each subspace of fixed evalue a — OK as long as countable # of (indep.) states for any a (e.g. in separable $\mathcal{H}$) [caution: this set spans space of eigenvectors, but *may not* be complete basis]

Completeness relation

If ϕ_i are a complete on basis for $\mathcal{H}$.

$$|\alpha\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\alpha\rangle \qquad \forall|\alpha\rangle\,,$$

so $\boxed{\sum_i |\phi_i\rangle\langle\phi_i| = \mathbb{1}}$ (*completeness*)
(sum of projections onto 1D subspaces)

Matrix and vector representations

If $\mathcal{H}$ is separable, $\exists$ a countable on basis, $|\phi_i\rangle$

can write $|\alpha\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\alpha\rangle \Rightarrow \begin{pmatrix} \langle\phi_1|\alpha\rangle \\ \langle\phi_2|\alpha\rangle \\ \vdots \end{pmatrix}$

$$\langle\beta| = \sum_i \langle\beta|\phi_i\rangle\langle\phi_i| \Rightarrow (\langle\beta|\phi_1\rangle\langle\beta|\phi_2\rangle\cdots)$$

$$A = \sum_{i,j} |\phi_i\rangle\langle\phi_i|A|\phi_j\rangle\langle\phi_j| \Rightarrow \begin{pmatrix} \langle\phi_1|A|\phi_1\rangle & \langle\phi_1|A|\phi_2\rangle & \cdots \\ \langle\phi_2|A|\phi_1\rangle & \langle\phi_2|A|\phi_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

If $\langle a_i|$ are a basis of O.N. eigenvectors w.r.t. A, $\langle a_i|A|a_j\rangle = a_i\delta_{ij}$

$$A = \sum |a_i\rangle a_i \langle a_i| \Rightarrow \begin{pmatrix} a_1 & & & \\ & a_2 & \bigcirc & \\ & \bigcirc & a_3 & \\ & & & \ddots \end{pmatrix}$$

Usual matrix interpretation of adjoint, dual correspondence

$$\langle\phi_i|A|\phi_j\rangle = \langle\phi_j|A^\dagger|\phi_i\rangle^* \qquad \text{(adjoint = conjugate transpose)}$$

dual: $|\alpha\rangle \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} \xrightarrow{\text{dual}} \langle\alpha| \Rightarrow (c_1^* \; c_2^* \; \cdots)$

$$\langle\alpha|\beta\rangle \Rightarrow \sum c_i^* d_i \qquad \text{inner product.}$$

$$(c_1^* \; c_2^* \; \cdots) \begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix}$$

When do eigenvectors of $A = A^\dagger$ form a complete basis for $\mathcal{H}$?

True when $\mathcal{H}$ is finite dimensional (explicit construction from diagonalization), not necessarily when $\mathcal{H}$ infinite dimensional.

Defs. A is bounded iff $\sup\limits_{\substack{|\alpha\rangle\in\mathcal{H} \\ |\alpha\rangle\neq|0\rangle}} \frac{\langle\alpha|A|\alpha\rangle}{\langle\alpha|\alpha\rangle} < \infty$

A is *compact* if every bounded sequence $\{|\alpha_n\rangle\}$ ($\langle\alpha_n|\alpha_n\rangle < \beta$) has a subsequence $\{|\alpha_{n_k}\rangle\}$ so that $\{A|\alpha_{n_k}\rangle\}$ is norm convergent in $\mathcal{H}$.

Facts:

- A compact $\Rightarrow$ A bounded.
- Every compact $A = A^\dagger$ has a complete set of eigenvectors. (compactness sufficient)
- not necessarily true for bounded $A = A^\dagger$. (neither necessary nor sufficient)

Ex. $\mathcal{H} = \mathcal{L}^2([0,1]) \qquad A = x.$

A is bounded, not compact. ($|\alpha_n\rangle = x^n\sqrt{2n+1}$)
A has *no eigenvectors* in $\mathcal{H}$.

For physics: Only interested in operators with a complete set of eigenvectors. These are called *observables*. Observables *need not* be bounded or compact. (note: will reverse this stance a bit for cts systems!)

Trace

The trace of an operator A is

$$\begin{aligned} \mathrm{Tr}\, A &= \sum_i \langle\phi_i|A|\phi_i\rangle\,, \qquad |\phi_i\rangle \text{ ON basis} \\ &= \sum_i a_i = \sum_i A_{ii}\,. \\ &\quad (a_i = \text{eigenvalues of } A) \end{aligned}$$

Unitary transformations

If $|a_i\rangle$, $|b_i\rangle$ are two complete ON bases, [(Ex. eigenkets of 2 Hermitian operators)]
can define U so that $U|a_i\rangle = |b_i\rangle$ (since $|a_i\rangle$ a basis defines U on all of $\mathcal{H}$).
so $\langle b_i| = \langle a_i|U^\dagger$.

We have

$$\begin{aligned} U = U\mathbb{1} &= U\textstyle\sum_i |a_i\rangle\langle a_i| \\ &= \textstyle\sum_i |b_i\rangle\langle a_i| \\ U^\dagger &= \textstyle\sum_i |a_i\rangle\langle b_i| \end{aligned}$$

so $UU^\dagger = \sum_{i,j}|b_i\rangle\langle a_i|a_j\rangle\langle b_j| = \delta_{ij}\sum_{ij}|b_i\rangle\langle b_j| = \mathbb{1}$ and $UU^\dagger = \mathbb{1}$, so $U^{-1} = U^+$, U unitary.

- Analogous to rotations in Euclidean 3-space $M : M^+M = MM^+ = \mathbb{1}$. U are symmetries of $\mathcal{H}$.

Unitary transforms of vectors & operators

A vector $|\alpha\rangle$ has representations in two bases as

$$|\alpha\rangle = \sum c_i|a_i\rangle = \sum d_i|b_i\rangle\,.$$

How are these related?

$$\begin{aligned} \sum d_j|b_j\rangle &= \sum_j d_j U|a_j\rangle \\ &= \sum_{i,j} d_j|a_i\rangle\langle a_i|U|a_j\rangle \end{aligned}$$

so $c_i = U_{ij}d_j$, $\qquad U_{ij} = \langle a_i|U|a_j\rangle$ are mtx elements of U in a rep.

Similarly, $X = |a_i\rangle X_{ij}\langle a_j| = |b_k\rangle Y_{k\ell}\langle b_\ell|$ gives $X_{ij} = U_{ik}Y_{k\ell}U^\dagger_{lj}$.

Diagonalization of Hermitian operators

Theorem. *A Hermitian matrix (finite dim) $H_{ij} = \langle\phi_i|H|\phi_j\rangle$ can always be diagonalized by a unitary transformation.*

Proof. if $|\phi_i\rangle$ a general ON basis, ON eigenvectors $|h_i\rangle$ related to $|\phi_i\rangle$ through

$$\begin{aligned} |h_i\rangle &= U|\phi_i\rangle\,, \qquad \text{unitary}\,. \\ \langle h_i|H|h_j\rangle &= \delta_{ij}h_i = \langle\phi_i|U^\dagger HU|\phi_j\rangle \end{aligned}$$

so $U^+_{ik}H_{k\ell}U_{\ell j}$ is diagonal.
(generalizes to any observable)

Algorithm for explicit diagonalization of a matrix H (finite dimensional):

1) Solve $\det(H - \lambda 1\!\!1) = 0$ for $N \times N$ matrices, N solutions are eigenvalues of H.

2) Solve $H_{ij}c_j = \lambda c_i$ for c_i's for each λ. N linear eqns. in N unknowns.

Gives eigenvalues & eigenvectors.

Invariants

Some functions of an operator A are invariant under U:

$$\begin{aligned} \operatorname{Tr} A &= \sum \langle\phi_i|A|\phi_i\rangle\,, \qquad |\phi_i\rangle \text{ ON basis} \\ \operatorname{Tr} U^\dagger AU &= \sum_{i,j,k} U^\dagger_{ij}A_{jk}U_{kj} = \delta_{jk}A_{jk} = \operatorname{Tr} A \end{aligned}$$

[Technical note: careful for ∞ matrices — need all sums converging.]

Another invariant: $\det A$: $\det(U^\dagger AU) = \det U \det A \det U^\dagger = \det UU^\dagger \det A = \det A$

[Note: full spectrum of ev's is invariant!]

Simultaneous diagonalization

Theorem. *Two diagonalizable operators A, B are simultaneously diagonalizable iff* $[A, B] = 0$

$$\begin{aligned} \Rightarrow \text{ say} \quad A|\alpha_i\rangle &= a_i|\alpha_i\rangle\,, \qquad B|\alpha_i\rangle = b_i|\alpha_i\rangle \\ AB|\alpha_i\rangle &= BA|\alpha_i\rangle = a_ib_i|\alpha_i\rangle\,. \end{aligned}$$

$$\begin{aligned} \Leftarrow \text{ Say} \quad AB = BA\,, \qquad &A|\alpha_i\rangle = a_i|\alpha_i\rangle\,. \\ AB|\alpha_i\rangle = a_iB|\alpha_i\rangle\,, \end{aligned}$$

so B keeps state in subspace of e.v. a_i. Thus, B is block-diagonal, can be diagonalized in each a_i subspace

□

1.3 The rules of quantum mechanics

[[Developed over many years in early part of C20. Cannot be derived — justified by logical consistency & agreement with experiment.]]

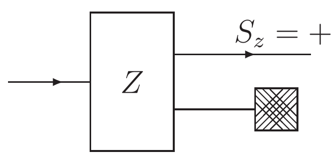
4 basic postulates:

1) A quantum system can be put into correspondence with a Hilbert space $\mathcal{H}$ so that a definite quantum state (at a fixed time t) corresponds to a definite ray in $\mathcal{H}$.

so $|\alpha\rangle \approx c|\alpha\rangle$ represent same physical state

convenient to choose $\langle\alpha|\alpha\rangle = 1$, leaving phase freedom $e^{i\phi}|\alpha\rangle$

- Note: still a classical picture of state space ("realist approach"). Path integral approach avoids this picture.
- "state" really should apply to an ensemble of identically prepared experiments ("pure ensemble" = pure state.)
  Ex. states coming out of SG filter



$$|\alpha\rangle = |+\rangle \ .$$

2) Observable quantities correspond to Hermitian operators whose eigenstates form a complete set.

Observable quantity = something you can measure in an experiment.

[[Note: book constructs $\mathcal{H}$ from eigenstates of A: logic less clear as $\mathcal{H}_A \neq \mathcal{H}_B$ for some A, B.]]

3) An observable $H = H^\dagger$ defines the time evolution of the state in $\mathcal{H}$ through

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = i\hbar \lim_{\Delta t\to 0}\frac{|\psi(t+\Delta t)\rangle - |\psi(t)\rangle}{\Delta t} = H|\psi(t)\rangle \, .$$

(Schrödinger equation)

4) (Measurement & collapse postulate)

If an observable A is measured when the system is in a normalized state $|\alpha\rangle$, where A has an ON basis of eigenvectors $|a_i\rangle$ with eigenvalues a_i.

a) The probability of observing $A = a$ is

$$\sum_{j:a_j=a} |\langle a_j|\alpha\rangle|^2 = \langle\alpha|P_a|\alpha\rangle$$

where $P_a = \sum_{j:a_j=a} |a_j\rangle\langle a_j|$ is the projector onto the $A = a$ eigenspace.

b) If $A = a$ is observed, after the measurement the state becomes $|\alpha_a\rangle = P_a|\alpha\rangle = \sum_{j:a_j=a} |a_j\rangle\langle a_j|\alpha\rangle$ (normalized state is $|\tilde{\alpha}_a\rangle = |\alpha_a\rangle/\sqrt{\langle\alpha_a|\alpha_a\rangle}$).

Discussion of rule (4):

Simplest case: nondegenerate eigenvalues

$$|\alpha\rangle = \sum c_i|a_i\rangle\,, \qquad a_i \neq a_j\,.$$

Then probability of getting $A = a_i$ is $|c_i|^2$.

Norm of $\langle\alpha|\alpha\rangle = 1 \qquad \Leftrightarrow \qquad \sum |c_i|^2 = 1$.

After measuring $A = a_i$, state becomes $|\tilde{\alpha}_i\rangle = |a_i\rangle$.

This postulate involves an *irreversible*, *nondeterministic*, and *discontinuous* change in the state of the system.

- source of considerable confusion
- less troublesome picture: path integrals.
- alternatives: non-local hidden variables ('t Hooft?), string theory — new principles (?)

For purposes of this course, take (4) as fundamental, though counterintuitive, postulate.

To discuss probabilities, need *ensembles*.

Consequence of (4):
Expectation value of an observable A in state $|\alpha\rangle$ is

$$\langle A\rangle = \sum_i |c_i|^2 a_i = \langle\alpha|A|\alpha\rangle \text{ since } A = \sum_i |a_i\rangle a_i\langle a_i|.$$

So for:

4 basic postulates of Quantum Mechanics:

1) State = ray in $\mathcal{H}$ [incl. def of H space]

2) Observable = Hermitian operator with complete set of eigenvectors

3) $i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$

4) Measurement & collapse

Probability $A = a$: $\langle\alpha|Pa|\alpha\rangle$

After measurement, system $\longrightarrow |\tilde{\alpha}_a\rangle = P_a|\alpha\rangle/\sqrt{\langle\alpha|P_a|\alpha\rangle}$

$$P_a = \sum_{j:a_j=a} |a_j\rangle\langle a_j|$$

$\big(\Rightarrow$ Expectation value of A: $\langle A\rangle = \langle\alpha|A|\alpha\rangle = \sum |c_i|^2 a_i$ if $|\alpha\rangle = \sum c_i|a_i\rangle\big)$

4 $\Rightarrow$

These are the rules of the game.

Rest of the course:

Examples of physical systems, tools to solve problems.

An example revisited in detail

Back to spin-$\frac{1}{2}$ system.

$$P1 \begin{cases} \text{State space} \\ \qquad \mathcal{H} = \{|\alpha\rangle = C_+|+\rangle + C_-|-\rangle\,, C_\pm \in \mathbb{C}\} \\ \text{Unit norm condition} \\ \qquad \langle\alpha|\alpha\rangle = |C_+|^2 + |C_-|^2 = 1 \\ e^{i\theta}|\alpha\rangle\,,\ |\alpha\rangle \text{ are physically equivalent} \end{cases}$$

$\longmapsto$

Operators:

$$\begin{aligned} S_z &= \tfrac{\hbar}{2}\sigma_z = \tfrac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && \text{measures spin along } z\text{-axis} \\ S_x &= \tfrac{\hbar}{2}\sigma_x = \tfrac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && \text{measures spin along } x\text{-axis} \\ S_y &= \tfrac{\hbar}{2}\sigma_y = \tfrac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && \text{measures spin along } y\text{-axis} \end{aligned}$$

For general axis $\underline{\hat{n}}$:

$$S_n = \underline{S} \cdot \underline{\hat{n}} \qquad \text{(HW \#2)}$$

has eigenvalues $\pm\frac{\hbar}{2}$.

eigenstates $|S_n;\pm\rangle$: $S_n|S_n;\pm\rangle = \pm\frac{\hbar}{2}|S_n;\pm\rangle$ form complete basis.

$$\left.\begin{array}{lcl} |S_x;\pm\rangle &=& \frac{1}{\sqrt{2}}|+\rangle \;\pm\; \frac{1}{\sqrt{2}}|-\rangle \\ |S_y;\pm\rangle &=& \frac{1}{\sqrt{2}}|+\rangle \;\pm\; \frac{i}{\sqrt{2}}|-\rangle \\ |S_z;\pm\rangle &=& |+\rangle \end{array}\right\} \text{in } S_z \text{ basis.}$$

Some further properties of S_i:

$$\begin{aligned} [S_i, S_j] &= i\epsilon_{ijk}\hbar S_k \\ \{S_i, S_j\} &= S_iS_j + S_jS_i = \tfrac{1}{2}\hbar^2\delta_{ij} \end{aligned}$$

$$S^2 = \underline{S}\cdot\underline{S} = S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2 \mathbb{1} = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And

$$[S^2, S_i] = 0\,.$$

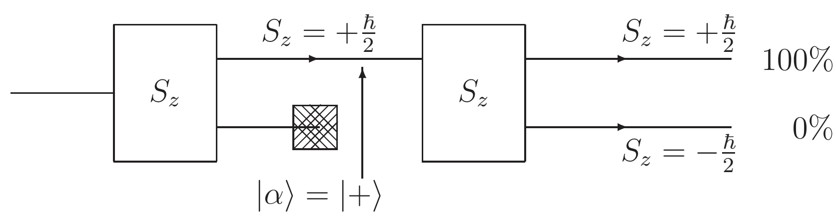
Measurement

If $|\alpha\rangle = c_+|+\rangle + c_-|-\rangle, \quad \langle\alpha|\alpha\rangle = 1,$

$$\begin{array}{lcccl} \text{prob. that } S_z &=& \frac{+\hbar}{2} & \text{is} & |c_+|^2 \\ \text{prob. that } S_z &=& \frac{-\hbar}{2} & \text{is} & |c_-|^2 \end{array}$$

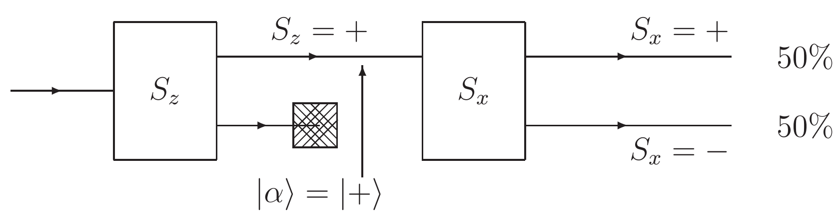
Consider single particle experiments from lecture 1.

i)



Repeated measurement of S_z gives $+\frac{\hbar}{2}$ 100% of the time.

ii)



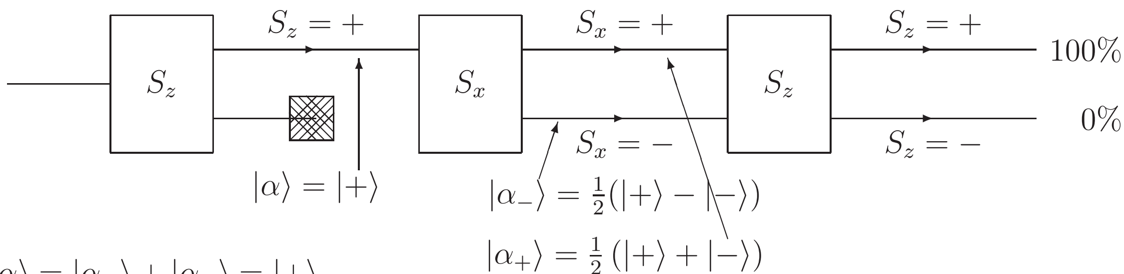
$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$

so $|\alpha\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left[|S_x;+\rangle + |S_x;-\rangle\right]$

so prob. $S_x = +\frac{\hbar}{2}$ is $\frac{1}{2}$ (50%)

prob. $S_x = -\frac{\hbar}{2}$ is $\frac{1}{2}$ (50%)

iii)

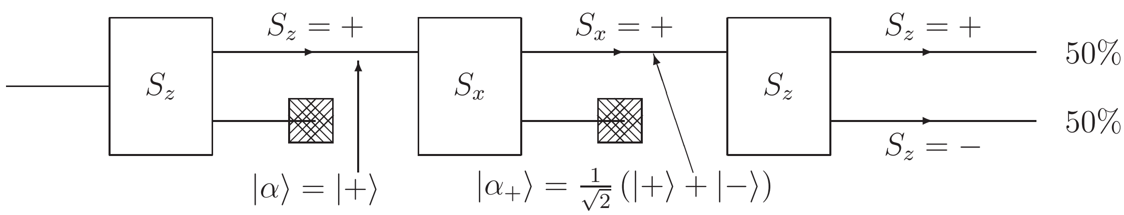


$|\alpha\rangle = |\alpha_+\rangle + |\alpha_-\rangle = |+\rangle$

Combined state $|\alpha\rangle = |+\rangle$ enters last measurement apparatus, since S_x not measured.

Gives $S_z = +\frac{\hbar}{2}$ 100% of time.

iv)



state $|\alpha_+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ enters last apparatus.

Prob. $S_x = +\frac{\hbar}{2}$: (50%)

Prob. $S_x = -\frac{\hbar}{2}$: (50%)

Compatible vs. incompatible observables

Observables A, B are:

Compatible if $[A, B] = AB - BA = 0$

incompatible if $[A, B] \neq 0$.

Examples: S^2, S_z are compatible
S_x, S_y are not compatible.

Theorem. *Compatible observables A, B can be simultaneously diagonalized, and have eigenvectors $|a_i, b_i\rangle$ with*

$$\begin{aligned} A|a_i, b_i\rangle &= a_i|a_i, b_i\rangle \\ B|a_i, b_i\rangle &= b_i|a_i, b_i\rangle \, . \end{aligned}$$

(Proof in last lecture: $AB|\alpha\rangle = aB|\alpha\rangle$ if $A|\alpha\rangle = a|\alpha\rangle$ so $B = \mathcal{H}_a \to \mathcal{H}_a$, diagonalize in each block.)

A *complete set of commuting observables* (CSCO) is a set of observables $\{A, B, C, \ldots\}$ such that all observables in the set commute:

$$[A,B] = [A,C] = [B,C] = \cdots = 0$$

and such that for any $a, b, \ldots$ at most one solution exists to the eigenvalue equations

$$\begin{aligned} A|\alpha\rangle &= a|\alpha\rangle \\ B|\alpha\rangle &= b|\alpha\rangle \\ &\vdots \end{aligned}$$

Tensor product spaces

$\big($useful for many-particle systems [+ quantum computing, ...]$bigr$)

Given two Hilbert spaces $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$, with complete ON bases $|\phi^{(1)}\rangle_i$, $|\phi^{(2)}\rangle_j$, the tensor product

$$\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$$

is the Hilbert space with ON basis

$$|\phi_{ij}\rangle = |\phi_i^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle$$

and inner product

$$\langle \phi_{i,j}|\phi_{k,\ell}\rangle = \langle \phi_i^{(1)}|\phi_k^{(1)}\rangle_1, \langle \phi_j^{(2)}|\phi_\ell^{(2)}\rangle_2 \,.$$

If $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ have dimensions N, M, then $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ has dimension NM.

If $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ separable, $\mathcal{H}$ is separable.

[in particular, if either or both of $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ have countable basis & both countable or finite $\mathcal{H}$ has countable basis.]

Tensor product of kets and operators

Kets:

If

$$\begin{aligned} |\alpha\rangle &= \sum c_i |\phi_i^{(1)}\rangle \in \mathcal{H}^{(1)} \\ |\beta\rangle &= \sum d_i |\phi_j^{(2)}\rangle \in \mathcal{H}^{(2)} \end{aligned}$$

are kets in $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$.

then

$$|\alpha\rangle \otimes |\beta\rangle = \sum_{i,j} c_i d_j |\phi_{i,j}\rangle \in \mathcal{H}$$

is in $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. [Note: not all vectors in $\mathcal{H}$ are of tensor product form. Ex. $|\phi_{1,1}\rangle + |\phi_{1,2}\rangle + |\phi_{2,1}\rangle$]

Operators:

If A, B are operators on $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$, then we can construct

$A \otimes B$ as an operator on $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ through

$$(A \otimes B)|\phi_{i,j}\rangle = (A|\phi_i^{(1)}\rangle) \otimes (B|\phi_j^{(2)}\rangle)\,.$$

[defines $A \otimes B$ on all of $\mathcal{H}$ by linearity]

If A, B are observables, then $A \otimes B$ is an observable.

Summary of Tensor product spaces

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \\
\dim \mathcal{H} &= (\dim \mathcal{H}^{(1)})(\dim \mathcal{H}^{(2)}) \\
\text{Basis: } |\phi_{i,j}\rangle &= |\phi^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle \\
\langle\phi_{i,j}\rangle &= \langle\phi^{(1)}| \otimes \langle\phi_j^{(2)}| \\
\text{Kets: } |\alpha\rangle \sum C_i |\phi_i^{(1)}\rangle |\beta\rangle &= \sum d_j |\phi_j^{(2)}\rangle \\
|\alpha\rangle \otimes |\beta\rangle &= \sum C_i d_j |\phi_{i,j}\rangle
\end{aligned}$$

Bras same

Operators $(A \otimes B) \sum C_{i,j} |\phi_{i,j}\rangle = \sum C_{i,j} (A|\phi_i^{(1)}\rangle) \otimes (B|\phi_j^{(2)}\rangle)$

Simple class of operators on $\mathcal{H}$:

$$A \otimes \mathbb{1}\,, \qquad \mathbb{1} \otimes B\,.$$

If A, B act on $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$, will often refer to these as just A, B when context is clear.

Useful relation:

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)\,.$$

Note: $[(A \otimes \mathbb{1}), (\mathbb{1} \otimes B)] = 0$.

Notation: in many books, tensor product symbol is omitted

$$\begin{aligned} |\alpha\rangle \otimes |\beta\rangle &\Rightarrow |\alpha\rangle|\beta\rangle \\ A \otimes B &\Rightarrow AB\,. \end{aligned}$$

CSCO's in tensor product spaces

If $\{A_1, A_2, \ldots, A_k\}$ are a CSCO for $\mathcal{H}^{(1)}$, & $\{B_1, \ldots, B_\ell\}$ are a CSCO for $\mathcal{H}^{(2)}$, then $\{A_1, \ldots, A_k, B_1, \ldots B_\ell\}$ are a CSCO for $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$

[[Ex. of notation $A_1 = A_1 \otimes \mathbb{1}$.]]

Example of tensor products:

Two spin-$\frac{1}{2}$ particles

Consider two spin-$\frac{1}{2}$ particles with Hilbert spaces $\mathcal{H}_2^{(1)}, \mathcal{H}_2^{(2)}$.

The two-particle Hilbert space is

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)}\,.$$

A basis for $\mathcal{H}$ is:

$$\begin{aligned} |{+}{+}\rangle &= |+\rangle_1 \otimes |+\rangle_2 \\ |{+}{-}\rangle &= |+\rangle_1 \otimes |-\rangle_2 \\ |{-}{+}\rangle &= |-\rangle_1 \otimes |+\rangle_2 \\ |{-}{-}\rangle &= |-\rangle_1 \otimes |-\rangle_2 \end{aligned}$$

Operators:

A complete set of commuting observables is

$$\begin{aligned} S_z^{(1)} &= S_z^{(1)} \otimes \mathbb{1} \\ S_z^{(1)} &= \mathbb{1} \otimes S_z^{(2)}\,. \end{aligned}$$

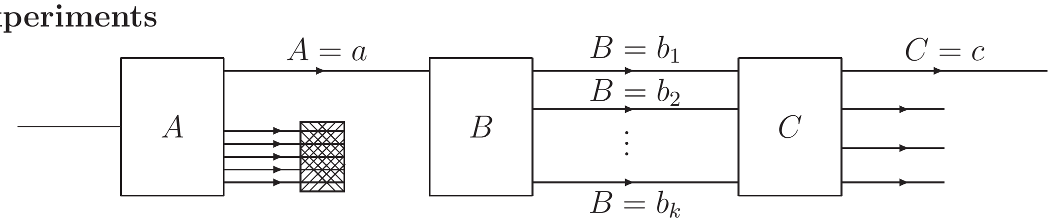
Consider operators

$$
\begin{aligned}
S_z &= S_z^{(1)} + S_z^{(2)}\,. \\
&= \begin{pmatrix} \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar \end{pmatrix} \\
S_z^{(1)} S_z^{(2)} &= (S^{(1)})_z \otimes \mathbb{1})(\mathbb{1} \otimes S^{(2)}) \\
&= S_z^{(1)} \otimes S_z^{(2)} \\
&= \begin{pmatrix} \frac{\hbar}{2} & & & \\ & \frac{\hbar}{2} & & \\ & & -\frac{\hbar}{2} & \\ & & & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} \frac{\hbar}{2} & & & \\ & -\frac{\hbar}{2} & & \\ & & \frac{\hbar}{2} & \\ & & & -\frac{\hbar}{2} \end{pmatrix} \\
&= \frac{\hbar^2}{4} \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}.
\end{aligned}
$$

Incompatible observables

If $[A, B] \neq 0$, then cannot simultaneously diagonize A, B.

Experiments



(Assume A, B, C nondegenerate)

i) Allow all b_i to combine without measuring B
Probability $(C = c) = |\langle c|a\rangle|^2$
[B not measured]

ii) Measure B_i & allow all parts to combine

$$\begin{aligned}
\text{Probability } (B = b_i) &= |\langle b_i|a\rangle|^2 \\
\text{Prob. } (C = c \text{ given } B = b_i) &= |\langle c|b_i\rangle|^2 \\
\text{Prob. } (C = c) &= \sum_i |\langle c|b_i\rangle|^2 \, |\langle b_i|a\rangle|^2 \\
\text{[when } B \text{ measured]} & \\
&= \sum_i \langle c|b_i\rangle\langle b_i|a\rangle\langle a|b_i\rangle\langle b_i|c\rangle \\
&= \sum_i z_i^* z_i \,, \qquad z_i = \langle a|b_i\rangle\langle b_i|c\rangle
\end{aligned}$$

know $(\sum z_i)(\sum z_i^*) = |\langle a|c\rangle|^2$.

So prob. $(C = c)$ does not depend on measurement of B when

$$\sum_i z_i^* z_i = (\sum_i z_i^*)(\sum_j z_j)\,.$$

Sufficient condition: only one $z_i \neq 0$,

so either $\langle a|b_i\rangle = 0$ or $\langle c|b_i\rangle = 0$ for all but one value of i

Sufficient condition: *either* $[A, B] = 0$ or $[B, C] = 0$.

Dispersion

For A an observable, $|\alpha\rangle$ a state,

define $\Delta A = A - \langle A\rangle$

$\langle \Delta A^2\rangle$ is *dispersion* of A.

$$\begin{aligned}
\langle \Delta A^2\rangle &= \langle A^2 - 2A\langle A\rangle + \langle A\rangle^2\rangle \\
&= \langle A^2\rangle - \langle A\rangle^2
\end{aligned}$$

is *variance* (a.k.a. mean square deviation) of A.

$$\begin{aligned}
\text{If } A|\alpha\rangle &= a|\alpha\rangle \\
\langle \Delta A^2\rangle &= a^2 - a^2 = 0
\end{aligned}$$

Variance measures "fuzziness" of state.

Example: In state $|+\rangle$

$$
\begin{aligned}
\langle \Delta S_z^2 \rangle &= \left[\begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}\frac{\hbar^2}{4} & \\ & \frac{\hbar^2}{4}\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}\right] \\
&= -\left[\begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}\frac{\hbar}{2} & \\ & -\frac{\hbar}{2}\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}\right]^2 \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0\,. \\
\langle \Delta S_z^2 \rangle &= \begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}\frac{\hbar^2}{4} & \\ & \frac{\hbar^2}{4}\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\
&= -\left[\begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}\right]^2 \\
&= \frac{\hbar^2}{4}\,.
\end{aligned}
$$

Uncertainty relation

If A, B are observables,

$$\langle \Delta A^2 \rangle \langle \Delta B \rangle^2 \geq \tfrac{1}{4} \left|[A,B]\right|^2$$

Proof.

$$
\begin{aligned}
\underset{(\text{on } \Delta A|\alpha\rangle, \Delta B|\alpha\rangle)}{\text{Schwartz:}} & \left(\langle\alpha|\Delta A)(\Delta A|\alpha\rangle\right)\left(\langle\alpha|\Delta B\rangle)(\Delta B|\alpha\rangle\right) \\
&\geq \left(\langle\alpha|\Delta A)(\Delta B|\alpha\rangle\right)\left(\langle\alpha|\Delta B)(\Delta A|\alpha\rangle\right) \\
&= \left|\langle\alpha| \left(\tfrac{1}{2}[\Delta A, \Delta B] + \tfrac{1}{2}\{\Delta A, \Delta B\}\right) |\alpha\rangle\right|^2 \\
&= \tfrac{1}{4}\Big| \underset{\substack{\uparrow \\ \text{imaginary} \\ ([A,B] \text{ skew-Hermitian}) \\ [\text{prob. 1–1.}]}}{\langle [A,B] \rangle} + \underset{\substack{\uparrow \\ \text{real} \\ (\{\Delta A, \Delta B\} \text{ Hermitian})}}{\langle \{\Delta A, \Delta B\} \rangle} \Big|^2 \\
&= \tfrac{1}{4} \left|\langle [A,B] \rangle\right|^2 + \tfrac{1}{4} \left|\langle \{\Delta A, \Delta B\} \rangle\right|^2 \\
&\geq \tfrac{1}{4} \left|\langle [A,B] \rangle\right|^2\,.
\end{aligned}
$$

□

Example: In state $|\alpha\rangle = |+\rangle$.

$$
\begin{aligned}
\langle \Delta S_z^2 \rangle &= 0 \\
\langle \Delta S_x^2 \rangle &= \frac{\hbar^2}{4}
\end{aligned}
$$

$$\tfrac{1}{4}\left|\langle [S_z, S_x]\rangle\right|^2 = \tfrac{1}{4}\left|\langle S_y\rangle\right|^2 = 0\,.$$

1.4 Position, momentum and translation

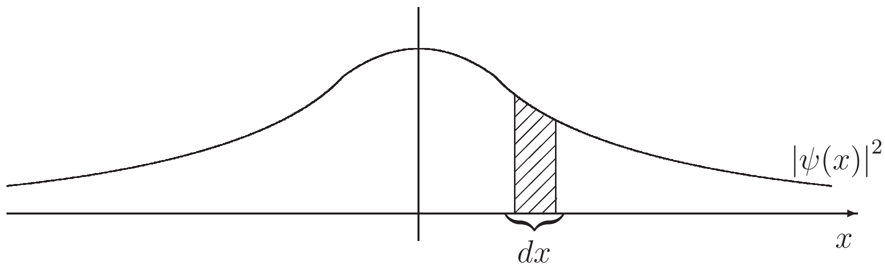
Until now, all explicit examples involved finite-dimensional matrices.

Generalize to *continuous degrees of freedom*.

Want to describe particle in 3D by wavefunction $\psi(x, y, z)$

Simply to 1D: $\psi(x)$

Want $|\psi(x)|^2\; dx =$ probability particle is in region dx.



Natural Hilbert space: $\mathcal{L}^2(\mathbb{R})$:
Square integrable functions $\int_{-\infty}^{\infty} |\psi(x)|^2 < \infty$.

[[To precisely define, need Lebesgue measure, ...]]

Can do QM in this framework.

$\mathcal{L}^2(\mathbb{R})$ is a separable Hilbert space.

Typical observables on $\mathcal{L}^{(2)}(\mathbb{R})$:

$$P_{[a,b]} \text{ projection on interval } [a, b]$$

$$(P_{[a,b]}f)(x) = \begin{cases} f(x)\,, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Quote from Von Neumann:

> Dirac, in several papers, as well as in his recently published book, has given a representation of quantum mechanics which is scarcely to be surpassed in brevity and elegance, and which is at the same time of invariant character. It is therefore perhaps fitting to advance a few arguments on behalf of our method, which deviates considerably from that of Dirac.
>
> The method of Dirac, mentioned above, (and this is overlooked today in a great part of quantum mechanical literature, because of the clarity and elegance of the theory)

in no way satisfies the requirements of mathematical rigor — not even if these are reduced in a natural and proper fashion to the extent common elsewhere in theoretical physics. For example, the method adheres to the fiction that each self-adjoint operator can be put in diagonal form. In the case of those operators for which this is not actually the case, this requires the introduction of "improper" functions with self-contradictory properties. The insertion of such a mathematical "fiction is frequently necessary in Dirac's approach, even though the problem at hand is merely one of calculating numerically the result of a clearly defined experiment. There would be no objection here if these concepts, which cannot be incorporated into the present day framework of analysis, were intrinsically necessary for the physical theory. Thus, as Newtonian mechanics first brought about the development of the infinitesimal calculus, which, in its original form, was undoubtedly not self-consistent, so quantum mechanics might suggest a new structure for our "analysis of infinitely many variables" — i.e., the mathematical technique would have to be changed, and not the physical theory. But this is by no means the case. It should rather be pointed out that the quantum mechanical "Transformation theory" can be established in a manner which is just as clear and unified, but which is also without mathematical objections. It should be emphasized that the correct structure need not consist in a mathematical refinement and explanation of the Dirac method, but rather that it requires a procedure differing from the very beginning, namely, the reliance on the Hilbert theory of operators.