8.311

Problem set #4

Prob 1 $\quad -\nabla\times E = \frac{1}{c}\partial B/\partial t \qquad \nabla\cdot E = \nabla\cdot B = 0$

$\nabla\times B = \frac{\varepsilon\mu}{c}\partial E/\partial t \qquad \vec{E}(z,t) = \vec{E}e^{ikz-i\omega t}$ (same for B)

$\rightarrow \vec{k}\times\vec{B} = -\frac{\varepsilon\mu}{c}\omega\vec{E}$, $\quad \vec{k}\times\vec{E} = \frac{1}{c}\omega\vec{B}$

$$\boxed{k^2 = \varepsilon\mu(\omega/c)^2 \qquad \vec{E}\cdot\vec{k} = \vec{B}\cdot\vec{k} = 0,\ \vec{B} = \sqrt{\varepsilon\mu}\,\hat{k}\times\vec{E}}$$

a) $\vec{E} = E_x\hat{x} + E_y\hat{y} \qquad B_x = -\sqrt{\varepsilon\mu}E_y,\ B_y = \sqrt{\varepsilon\mu}E_x$

$\vec{B} = B_x\hat{x} + B_y\hat{y}$

Linear polarization E_x/E_y real

circular polariz. $E_x/E_y = \pm i$

b) (i) $E_x^{(1)}/E_y^{(2)}$ real $\rightarrow E_1 + E_2$ linearly polarized

(ii) $E_x^{(1)}/E_2^{(2)} = \pm i \rightarrow E_1 + E_2$ circular polarized

Prob 2 $\quad$ a) $\omega' = \omega,\ k'_{\parallel} = k_{\parallel} \rightarrow k' = nk,\ k'\sin\theta' = k\sin\theta \rightarrow \underline{\sin\theta = n\sin\theta'}$

b) continuity at boundary:

$E_{\parallel}$: $E_0 + E_r = E_t$

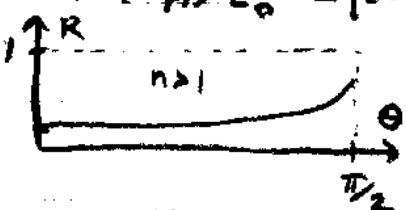
$B_{\parallel}$: $B_0\cos\theta - B_r\cos\theta = B_t\cos\theta'$

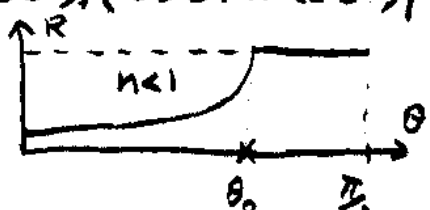
$\rightarrow \quad E_0 + E_r = E_t$

$E_0 - E_r = E_t\, n\cos\theta'/\cos\theta$

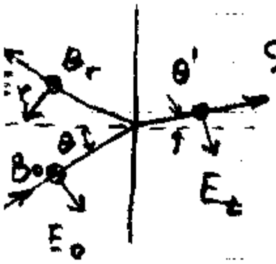
$$E_t = \frac{2\cos\theta\, E_0}{n\cos\theta' + \cos\theta} \qquad E_r = \frac{\cos\theta - n\cos\theta'}{\cos\theta + n\cos\theta'}E_0$$

$$R = |E_r|^2/E_0^2 = \left|(\cos\theta - n\cos\theta')/(\cos\theta + n\cos\theta')\right|^2$$





$\underline{\sin\theta_0 = n}$



c) continuity at boundary

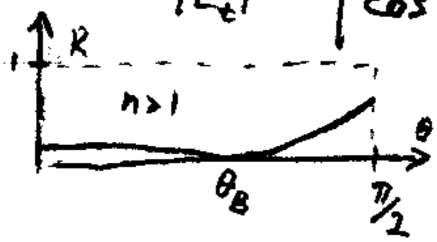
$E_{\parallel}$: $E_0\cos\theta + E_r\cos\theta = E_t\cos\theta'$

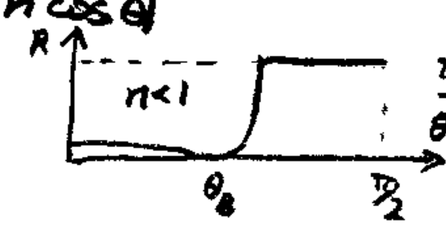
$D_{\perp}$: $E_0\sin\theta - E_r\sin\theta = \varepsilon E_t\sin\theta'$

$$\rightarrow \quad E_t = \frac{2\cos\theta}{n\cos\theta + \cos\theta'}E_0 \qquad E_r = \frac{\cos\theta' - n\cos\theta}{\cos\theta' + n\cos\theta}E_0$$

$$R(\theta) = \frac{|E_r|^2}{|E_t|^2} = \left|\frac{\cos\theta' - n\cos\theta}{\cos\theta' + n\cos\theta}\right|^2 \rightarrow R = 0 \text{ for } \theta + \theta' = \pi/2$$

$\underline{\tan\theta_B = n}$ Brewster angle





Prob 3 a) $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$ $\quad k^2 = \frac{\omega^2}{c^2}\varepsilon = \frac{\omega^2 - \omega_p^2}{c^2}$

$\omega(k) = (\omega_p^2 + c^2 k^2)^{1/2} \geq \omega_p \rightarrow$ no waves at $\omega < \omega_p$

b) $n = \sqrt{\varepsilon} < 1$ $(\omega > \omega_p) \rightarrow$ total reflection for $\sin\theta > n$
(see Prob 2 a)

c) $\tan\theta = \frac{L}{2h} = \frac{5}{3}$ $\quad \omega_p = \omega\cos\theta \quad \omega_p^2 = 4\pi Ne^2/m$

$n = \left(1 - \omega_p^2/\omega^2\right)^{1/2} = \sin\theta$ $\quad N = \frac{m}{4\pi e^2}\left(\frac{2\pi c}{\lambda}\right)^2 \times 0.265$

$$N = \frac{9\cdot10^{-28}\cdot(3\cdot10^{10})^2\,\pi\cdot 0.265}{(4.8\cdot10^{-10})^2\,(2.1\cdot10^3)^2} = 6\cdot10^5\ \text{cm}^{-3}$$ (between the day and night values)

Prob 4 $\quad \nabla\times B = \frac{4\pi}{c}j - \frac{i\omega}{c}E = -\frac{i\omega}{c}\left(1 + i\frac{4\pi\sigma}{\omega}\right)E$ $\quad \varepsilon_{eff} = 1 + i\frac{4\pi\sigma}{\omega}$

a) $k = \sqrt{\varepsilon}\,\omega/c = \frac{1+i}{\sqrt{2}}\frac{\sqrt{4\pi\sigma\omega}}{c}$ $\quad (\omega_{opt} \ll \sigma)$

$|E|^2 \propto e^{-2z/\ell}$, $\quad \ell = \frac{c}{\sqrt{2\pi\sigma\omega}} = \frac{\lambda}{2\pi\sqrt{2\pi\sigma/\omega}}$

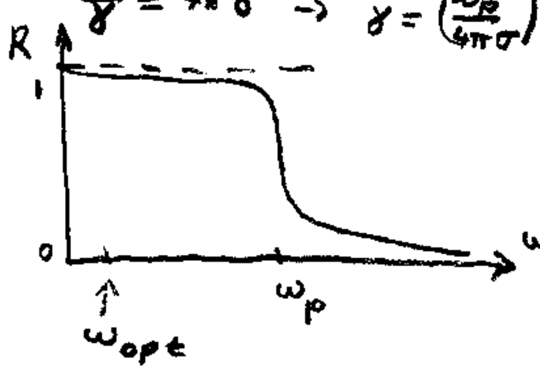
$\lambda = 0.5\mu m \quad \nu = \frac{\omega}{2\pi} = c/\lambda = \frac{3\cdot10^8}{0.5\cdot10^{-6}} = 6\cdot10^{14}$ Hz

$\ell = \frac{\lambda}{81.6} = \underline{6\ nm}$

b) $\omega_p = (4\pi Ne^2/m)^{1/2} = \left(4\pi\cdot10^{23}\cdot4.8^2\cdot10^{-20}/(9.1\cdot10^{-28})\right)^{1/2} = 1.78\cdot10^{16}$ Hz

$R(\omega) = \left|\frac{1-n}{1+n}\right|^2 \quad n = \sqrt{1 - \frac{\omega_p^2}{\omega(\omega - i\gamma)}}$

$\frac{\omega_p^2}{\gamma} = 4\pi\sigma \rightarrow \gamma = \left(\frac{\omega_p}{4\pi\sigma}\right)\omega_p = 1.4\cdot10^{-2}\omega_p$



$\omega_{opt} \approx 3.6\cdot10^{15}\ s^{-1} \sim 0.2\ \omega_p$

$\underline{R \approx 0.972}$

Prob 5 a) Monochromatic plane wave in 1d

$E, B \propto \exp(\pm ikx - i\omega t) \qquad k = n\,\omega/c$

General solution:

$$u(x,t) = \int \left(A(\omega)e^{ik(\omega)x} + B(\omega)e^{-ik(\omega)x}\right)e^{-i\omega t}\frac{d\omega}{2\pi}$$

b) $A = A_0 e^{-\tau^2(\omega-\omega_0)^2/2}, \quad B = 0$ use integra[l]

$$F(t) \equiv u(t)_{x=0} = e^{-i\omega_0 t}\, e^{-t^2/2\tau^2}(2\pi)^{-1/2}A_0 \qquad \sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2}dx$$

c) $k(\omega) = k(\omega_0) + \frac{1}{V_g}(\omega-\omega_0) + O(\delta\omega^2)$

$$u(x,t) = \int A(\omega)\, e^{ik_0x - i\omega_0 t}\, e^{i(x/v_g - t)(\omega-\omega_0)}\frac{d\omega}{2\pi}$$

$$= e^{-i\omega_0 t + ik_0 x}\, F(t - x/v_g) = (\text{phase factor}) \times (\text{envelop[e]})$$

Prob 6 a) $\vec{B} = B_0\hat{y}\, e^{ikx}\begin{cases} e^{-k_1 z}, & z>0 \\ e^{k_2 z}, & z<0 \end{cases}$

$$(\nabla\times B)_x = \left(\frac{1}{c}\frac{\partial D}{\partial t}\right)_x = -\frac{i\omega}{c}\hat{x}\, e^{ikx}\begin{cases} \varepsilon_1 E_x e^{-k_1 z} & z>0 \\ \varepsilon_2 E_x e^{k_2 z} & z<0 \end{cases}$$

$E_n, B_\parallel$ continuous at interface, thus

$$k_1/\varepsilon_1 = -k_2/\varepsilon_2 \;\rightarrow\; \left(k^2 - \frac{\omega^2}{c^2}\varepsilon_1\right)^{1/2} = \frac{\varepsilon_1}{|\varepsilon_2|}\left(k^2 + \frac{\omega^2}{c^2}|\varepsilon_2|\right)^{1/2}$$

$$\frac{\omega^2}{c^2}\left(\varepsilon_1 + \frac{\varepsilon_1^2}{|\varepsilon_2|}\right) = k^2\left(1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}\right) \rightarrow \text{for RHS} > 0 \text{ must hav[e]}$$

$$|\varepsilon_2| > \varepsilon_1$$

b) $\varepsilon_1 = 1, \; \varepsilon_2 = 1 - \frac{\omega_p^2}{\omega^2} < 0, \; \omega < \omega_p$

$$\frac{\omega^2}{c^2}\left(1 + \frac{1}{\frac{\omega_p^2}{\omega^2} - 1}\right) = k^2\left(1 - \frac{1}{\left(\frac{\omega_p^2}{\omega^2}-1\right)^2}\right) \rightarrow \frac{\omega^2}{c^2k^2} = 1 - \frac{\omega^2}{\omega_p^2 - \omega^2}$$

$$\underline{c^2k^2 = \frac{\omega^2(\omega_p^2 - \omega^2)}{\omega_p^2 - 2\omega^2}}$$

Surface EM wave dispersion relation