

8.311: Electromagnetic Theory Problem Set # 6 Due: 3/17/04

EM in confined geometry: resonance cavities, transmission lines

Reading: Schwinger, Chap. 44 (or Jackson, Chap. 2,8)

Variable separation: In the problems formulated in terms of electrostatic potential, one often has to analyze Poisson equation $\nabla^2\Phi = -4\pi\rho$ with appropriate boundary conditions on a surface of a conductor or a dielectric. There is a popular approach to this problem which starts with constructing a complete set of solutions of the Laplace's equation $\nabla^2\Phi = 0$ in the same geometry, and then using it as a basis in the space of functions in order to represent $\Phi(\mathbf{r})$ in question. Out of many geometry types amenable to the variable separation technique, the three most important ones are the *rectangular, spherical, and cylindrical* cases. In Problem 1 we review the main ideas of this approach for rectangular geometry using material from electrostatics, and then proceed with applying it to EM waves in resonant cavities and transmission lines.

1. Variable separation in electrostatics

Consider electric field in a rectangular slit $0 < x < a$, $0 < y < \infty$, with the boundaries $x = 0, a$ being ideal conductors.

a) Suppose that the boundary $y = 0$ is also an ideal conductor at a potential $\Phi = V$, while the sides $x = 0, a$ are grounded, $\Phi = 0$. Write down the potential inside the slit as a sum of terms corresponding to standing waves across the slit,

$$\Phi(x, y) = \sum_{n>0} \phi_n(y) \sin(k_n x), \quad k_n = \pi n/a \tag{1}$$

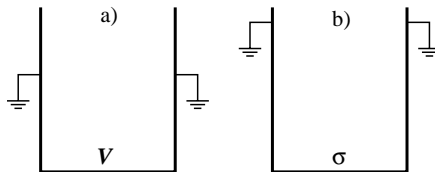
Using the orthogonality property of the standing waves

$$\int_0^a \sin(k_n x) \sin(k_m x) dx = (a/2)\delta_{mn} \tag{2}$$

and their completeness, show that any function of x and y can be represented in the form (1).

Show that the Laplace's equation for $\Phi(x, y)$ yields ordinary differential equations for the coefficients $\phi_n(y)$. By solving those equations with appropriate boundary conditions, obtain the potential $\Phi(x, y)$.

b) Now, suppose that the boundary $y = 0$ carries a frozen constant charge density σ , while the other two boundaries are ideal grounded conductors as above. Solve the problem for Φ by generalizing/modifying the approach of part a).



c) Consider a cube $a \times a \times a$ with ideal conducting boundary. One face of a cube is held at a potential V , while the other five faces are grounded. Using Fourier series similar to (1), find the potential $\Phi(\mathbf{r})$ within the cube. Determine the value Φ at the cube center (numerically or analytically).

2. Boundary condition for a nearly-perfect conductor

Consider a flat surface of a ohmic conductor in the presence of electromagnetic field of frequency $\omega \ll \sigma$, the conductivity. Show that near the surface the components of the fields \mathbf{E} and \mathbf{B} are related by

$$\mathbf{E}_{\parallel} = \zeta \mathbf{B}_{\parallel} \times \mathbf{n}, \quad \zeta = (1 - i) \sqrt{\frac{\omega}{8\pi\sigma}} \quad (3)$$

where \mathbf{n} is the normal vector pointing in the conductor. This is the boundary condition on the conductor surface for electromagnetic waves written in terms of so-called *surface impedance* ζ .

In the perfect conductor limit, $\omega/\sigma \rightarrow 0$, we have $\zeta = 0$. Show that in this case the field components obey the boundary conditions $\mathbf{E}_{\parallel} = 0$, $\mathbf{B}_{\perp} = 0$.

3. EM resonances of a rectangular cavity

Consider a rectangular cavity $a \times b \times c$ with boundaries being ideal conductors. To find resonance frequencies, determine when the cavity can hold an EM field with a harmonic time dependence, $\mathbf{E}, \mathbf{B} \propto e^{-i\omega t}$. The field obeys the wave equation in the domain $0 < x < a$, $0 < y < b$, $0 < z < c$,

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2} \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = 0 \quad (4)$$

inside the cavity (similar for \mathbf{B}), with the boundary conditions $\mathbf{E}_{\parallel} = 0$, $\mathbf{B}_{\perp} = 0$ at the surface.

a) Show that a family of solutions to this problem can be constructed using products of standing waves of the form

$$E_x(\mathbf{r}) = E_x^{(0)} \cos(k_x x) \sin(k_y y) \sin(k_z z), \quad k_n = \pi n / \ell \quad (5)$$

and similar for other components of \mathbf{E} , where $\ell = a, b, c$ for $k_{x(y,z)}$. Find the magnetic field using Faraday's law, and verify that the boundary conditions are fulfilled. Find the relation between different components of the electric field.

b) From the above, determine the spectrum of the resonance frequencies of the system.

4. Properties of EM resonances

a) Consider a resonance mode in a cavity of an arbitrary shape. Show that the time-averaged electric and magnetic field energies

$$\int \frac{\mathbf{E}^2}{8\pi} d^3 \mathbf{r}, \quad \int \frac{\mathbf{B}^2}{8\pi} d^3 \mathbf{r} \quad (6)$$

are equal. (Hint: starting from $(\omega^2/c^2) \int (\mathbf{E}^2/8\pi)(d\mathbf{r})$, use the wave equation (4) to express $\omega^2 \mathbf{E}/c^2$ in terms of the Laplacian of \mathbf{E} , and relate $\nabla \times \mathbf{E}$ with \mathbf{B} .)

b) For two EM resonance modes with different frequencies $\omega_1 \neq \omega_2$ prove the orthogonality property: $\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3 \mathbf{r} = 0$, $\int \mathbf{B}_1 \cdot \mathbf{B}_2 d^3 \mathbf{r} = 0$. (Hint: use the same method as in part a.)

5. Rectangular waveguide

Consider electromagnetic waves in a waveguide of a rectangular cross-section $a \times b$ with perfectly conducting walls. There are two kinds of waves: TE and TM. The TE (TM) modes have electric (magnetic) field transverse to \mathbf{k} , respectively.

a) Show that:

(i) The TE modes are given by the solutions to the boundary value problem for the magnetic field

component parallel to the wavevector, $(\nabla_{\perp}^2 + (\omega^2/c^2 - k^2))B_{\parallel} = 0$, with the boundary condition $\partial B_{\parallel}/\partial n = 0$.

(ii) The TM modes are given by the solutions of the boundary value problem for the parallel component of electric field, $(\nabla_{\perp}^2 + (\omega^2/c^2 - k^2))E_{\parallel} = 0$, with the boundary condition $E_{\parallel} = 0$.

b) Find the TE and TM modes for this problem. Give expressions for the cutoff frequencies.

6. Coaxial transmission line

Consider a coaxial cable made of a long cylindrical conducting shell of inner radius a and a wire of radius b on the axis inside. The cable is used as a transmission line in the operating (TEM) mode, for which both the electric and magnetic field are transverse to the wavevector.

a) Find the fields \mathbf{E} , \mathbf{B} for this mode and show that the dispersion relation for this mode is $\omega = ck$, as in vacuum.

b) Now, the cable of length L is made a part of an RC circuit, as shown in the figure. In this case the cable represents an extended capacitance C . Consider how the transmission line is being charged after the switch is closed and the battery voltage V is applied to the cable end. Closing the switch creates a step-like EM pulse of strength V which propagates from the battery end to the resistor, where it is partially reflected back, then again reflected from the battery end, and so on. The process can be understood using the results derived for the 1d wave equation (HW3, Problem 3), with appropriate boundary conditions describing the field at the ends of the cable.

Find the potential on the load resistor R as a function of time. Show that the time dependence can be described as ringing with frequency $c/2L$ (compare to the case of a small size capacitor which can be charged instantly).

Consider the ringing time dependence separately the cases when the load resistance R is large and small compared to the inverse light speed $1/c$. At what value of R the ringing is absent? (This is called an impedance-matched load/cable system.)

