PROFESSOR: OK, in that case I can begin by giving a quick review of last time. We began last time by talking about the data of measurements of the cosmic microwave background, starting with the data as it existed in 1975 which I advertised as being an incredible mess, which it was. You could easily believe that this data fit this solid line curve, which was what it's supposed to fit. But you could equally well believe that it didn't. Things got worse before they got better. There was the famous Berkeley-Nagoya Rocket Flight experiment of 1987 which had a data point which missed the theoretical curve by 16 standard deviations, which might seem fairly disappointing.

It reminds one, by the way, of a famous quote of Arthur Eddington-- which you may or may not be familiar with-- but Eddington pointed out that while we always say that we should not believe theories until the confirmed by experiment, it's in fact equally true that we should not believe data that's put forward until it's confirmed by theory, and that certainly was the case here. This data was never confirmed by theory and turned out to be wrong. The beautiful data was achieved in 1990 by this fabulous COBE satellite experiment, which showed-- unambiguously, for the first time-- that the cosmic background radiation really does obey an essentially perfect blackbody curve, which is just gorgeous.

We then went on, in our last lecture, to begin to talk about the cosmological constant and its effect on the evolution of the universe-- completely changing gear here-- and the key issue is the cosmological effect of pressure. Earlier we had derived this equation. The equation shows the significant role of pressure during the radiation-dominated era, but it also shows that pressure-- if it were negative-- could perhaps have the opposite effect, causing an acceleration of the universe.

Furthermore, we learned last time that vacuum energy-- first thing we learned, I guess, is just that's synonymous with Einstein's cosmological constant, related to Einstein's cosmological constant lambda by this equation. That is, the energy density of the vacuum is equal to the mass density of the vacuum times C squared and is equal to this expression in terms of Einstein's original cosmological constant. And most important, in terms of physics, we learned that if we have a non-zero vacuum energy-- vacuum energy by definition does not change with time because the vacuum is the vacuum, it's simply the lowest possible energy density allowed by the laws of physics, and the laws of physics, as far as we know, do not change with time, and therefore vacuum energy density does not change with time-- and that is enough to imply that the pressure of the vacuum has to be equal to minus the energy density in a vacuum, and therefore minus that expression in terms of the cosmological constant-- which is exactly what will give us a repulsive gravitational effect, where we put that into the Friedmann equation.

Now I should emphasize here that the effect of the pressure that we are talking about is not the mechanical effect of the pressure. The mechanical effect of the pressure is literally zero because the pressure that we are discussing here is a uniform pressure, and pressures only cause mechanical forces when there's gradients in the pressure-- more pressure on one side than the other. And if all this pressure is always balanced, the mechanical force of the pressure is zero. But nonetheless, that pressure contributes-- according to Einstein's equations-- as a contribution to the gravitational field, and a positive pressure creates an attractive gravitational field but a negative pressure produces repulsive gravitational fields. And a positive vacuum energy corresponds to a negative pressure which, in fact, would dominate this equation, resulting in a gravitational repulsion.

So it's useful to divide the total energy density into a normal component plus vacuum energy, and similarly we can divide the pressure into a normal component plus the vacuum contribution to the pressure. The vacuum contribution to the pressure will instantly disappear from all of our equations because we know how to express it in terms of the vacuum mass density. It's just minus Rho vac C squared. So we can then re-write the Friedmann equations making those substitutions, and we conceive-- in the second-order equation-- that the vacuum contribution, negative, negative is a positive, produces a positive acceleration-- as we've been saying-- and a positive vacuum energy also contributes to the right-hand side of the first-order Friedmann equation. And in many, many situations-- although not quite all-- this vacuum energy will dominate at late times.

It definitely falls off more slowly than any of the other contributions. The vacuum energy is a constant, and every other contribution to that right-hand side falls off with A. The only way that Rho vac can fail to dominate if it's non-zero is if you have a closed universe that collapses before it has a chance to dominate, which is a possibility. But barring that, eventually the vacuum energy will dominate, and once the vacuum energy dominates, we just have H squared-- A dot over A is H. H squared equals a constant, and that just says that H approaches its vacuum value, which is the square root of 8 pi over 3 G Rho vac, and with H being a constant we can also solve for A. The scale factor itself is just proportional to an exponential of E to the H T, where H is the H associated with the vacuum.

So this will be a very easy to obtain asymptotic solution to the equations of the universe, and, in fact, we think that our real universe is approaching an exponentially expanding phase of exactly this sort today. We're not there yet, but we are approaching it. Yes?

AUDIENCE: So does an exponential A of T mean that the universe just keeps expanding forever and just spins out into nothingness?

PROFESSOR: Yes, an exponentially expanding A means the universe will continue expanding forever and ordinary matter will thin out to nothing, but this vacuum energy density will remain as a constant contribution. So the universe would go on expanding exponentially forever. Now, there is the possibility that this vacuum that we're living in is actually what might be called a false vacuum-- that is, an unstable vacuum-- a vacuum which behaves like a vacuum for a long time but is subjected to the possibility of decay. If that's the case, it's still true that most of our future space time will continue to exponentially expand exactly as this equation shows, but a kind of a Swiss cheese situation will develop where decays in the vacuum would occur in places, producing spherical holes in this otherwise exponentially expanding background. We'll be talking a little bit more about that later in the course. Any other questions?

OK, well what I want to do now is to work on a few calculations which I'd like to present today. If all goes well, we might have as many as three calculations, or at least two calculations that we'll do and one that we'll talk about a little bit. The first thing I want to do-- and I guess we just talked about starting it last time-- is calculate the age of the universe in this context. How do we express the age of the universe in terms of measurable, cosmological parameters, taking into account the fact that vacuum energy is one of the ingredients of our universe along with radiation and non-relativistic matter, which we have already discussed.

So to start that calculation we can write down the first-order Friedmann equation. A dot over A squared is equal to 8 pi over 3 G times Rho and now I'm going to divide Rho into all contributions we know about. Rho sub M, which represents non-relativistic matter, plus Rho sub radiation, which represents radiation, plus vacuum energy, which is our new contribution, which will not depend on time at all. And then to complete the equation there is minus KC squared over A squared. And the strategy here is really simply that because we know the time evolution of each of the terms on the right-hand side, we will be able to start from wherever we are today in the universe-- which will just take from data, the values of these mass densities-- and we will be able to integrate backwards and ask how far back do we have to go before we find the time when the scale factor vanished, which is the instant of the Big Bang.

So what we want to do is to put into this equation the time dependents that we know. So Rho sub M of T, for example, can be written as A of T naught divided by A of T cubed, times Rho sub M zero. And all these zeroes, of course, mean the present time. So this formula says, first of all, that the mass density falls off as 1 over the cube of the scale factor. A of T is the only factor on the right-hand here that depends on T. The numerator depends on T sub zero, but not T. The other constant, T is zero in the numerator and Rho sub M zero. Rho sub M zero denotes the present value of the mass density. And the constants here are just rearranged so that when T equals T naught, you just get Rho is equal to its present value.

And we can do the same thing for radiation, and here I won't write everything out because most things are the same. The quantity in brackets will be the same but this time it will occur to the fourth power because radiation falls off like the fourth power of the matter-- fourth power of the scale factor-- and then we have Rho radiation zero. And then finally, for the vacuum energy, we will just write on the blackboard what we already know, which is that's independent of time. So this gives us the time dependents of all three terms here.

Now we could just go from there, but cosmologists like to talk about mass densities in terms of the fraction of the critical density, omega. So we're going to change the notation just to correspond to the way people usually talk about these things. So we will recall that the critical density-- which is just that total mass density that makes little K equal to zero and hence the universe geometrically flat-- so Rho sub C, we learned, is 3 H squared over 8 pi G and then we will introduce different components of omega.

So I'm going to write omega sub X here where X really is just a stand-in for matter or radiation or vacuum energy. And whichever one of those we're talking about, omega sub X is just a shorthand for the corresponding mass density, but normalized by dividing by this critical density. And then I'm just going to rewrite these three equations in terms of omega instead of Rho. So Rho sub M of T becomes then 3 H naught squared over 8 pi G times the same A of T zero over A of T cubed, but not I'm multiplying, omega sub M zero. And from the definitions we've just written, this equation is just a rewriting of that equation.

And we can do the same thing, of course, for radiation. Rho radiation of T is equal to the same factor out front, the same quantity in brackets but this time to the fourth power, and then times omega radiation at the present time. And finally, Rho vac doesn't really depend on time-- but we'll write it as if it was a function of time-- it consists of the same factor of 3 H naught squared over 8 pi G, and no powers of the quantity in brackets, but then just multiplying omega sub vac zero.

[ELECTRONIC RINGING]

Everybody should turn off their cell phone, by the way.

[LAUGHING]

OK, sorry about that.

OK, now to make the equation look prettier, I'm going to rewrite even this last term as if it has something to do with an omega. And we can do that by defining omega sub K zero to be equal to minus KC squared over A squared of T naught times H naught squared, which is just the last term that appears there [INAUDIBLE] of factor of H squared, which we'll be able to factor out. And doing all that, the original Friedmann equation can now be rewritten as H squared-- also known as A dot over A squared-- can be written as H naught squared-- oh, I'm sorry, one other definition I want to introduce here. This ratio-- A of T naught over A-- keeps recurring, so it's nice to give it a name, and I'm going to give 1 over that a name. I'm going to let X equal the scale factor normalized by the scale factor today.

And I might point out that in Barbara Ryden's book, what I'm calling X is just what she calls the scale factor, because she chooses to normalize the scale factor so that it's equal to 1 today. So we haven't done that yet but we are effectively doing it here by redefining a new thing X. Having done that, the right-hand side of the Friedmann equation can now be written in a simple way. It's H naught squared over X squared times a function F of X-- which is just an abbreviation to not have to write something many times-- this is not, by any means, a standard definition. It really is just for today. It allows us to save some writing on the blackboard.

So I'm going to, for today, be using the abbreviation F of X is equal to omega sub M zero times X plus omega sub radiation zero times no powers of X plus omega sub vac zero times X to the fourth, and finally plus omega sub K zero times X squared. And this just lists all the things that would occur in parentheses here if we factored everything out.

Notice I factored out some powers of X squared, so the powers of X that appear here do not look familiar, but the relative powers do. That is, omega should fall like-- omega matter should fall like 1 over X cubed. Omega radiation should fall off like one power faster than that, and it does. This is one less power of X there than there, and omega vacuum should fall off like four powers of X different from radiation, and it does, et cetera. But there's no real offset here that makes the factors there not look familiar.

OK, all of this was just to put things in a simple form, but there's one other very useful fact to look at. Suppose we now apply this for T equals T zero, which means for X equals 1. OK, it's true at any time, but in particular we can look at what it says for X equals 1 and it tells us something about our definitions that we could have noticed in other ways-- but didn't notice yet-- which is that we set X equal to 1 here. These just becomes the sum of omegas. The powers of X's all become just ones. And the left-hand side is just H zero squared, which matches the H zero squared here because at T equals T naught H squared is H zero squared, so these H zeros squareds cancel.

So applying it to T equals T naught X equal to 1, what you get is simply 1 is equal to omega sum M zero plus omega sub radiation zero plus omega sub vac zero plus omega sub K zero, which gives us the simplest way of thinking about what this omega sub K zero really means. We defined it initially up there in terms of little K, but for this equation, we can see that omega sub K zero really is just another way of writing 1 minus all the other omegas.

So you could think of this as being the definition of what I'm calling omega sub k zero. So it's a language in which you essentially think that the total omega has to equal 1, and whatever is not contained in real matter becomes a piece of omega sub k, the curvature or contribution to omega.

OK, now it's really just a matter of simple manipulations and I-- the main purpose of defining F of x is to be able to write these simple manipulations simpler than they would be if you had to write out what F of x was every time. We're first just going to take the square root of the key equation up there-- the Friedmann equation-- and we get a dot over a is also equal to x dot over x. Note that the constant of proportionality there, a of t naught-- which is a constant-- cancels when you take a dot over a. So a dot over a is the same as x dot over x. And that-- just taking the square root of that equation-- is H naught over x. Hold on a second.

Yeah, we're taking the square root of the equation, so yeah, we had H naught squared over x squared. Here we have H naught over x and then just times the square root of f. And these x's cancel each other. Wait a minute. Oh, I'm sorry. They're not supposed to cancel because I didn't write this quite correctly. That should have been x to the fourth. Apologies. And now here we have x squared. And this can just be-- by manipulating where the x's go-- rewritten as x times dx dt. So I multiply the whole equation by x squared to get rid of that factor on the right, and now on the right-hand side we just have H zero times square root of F.

And now I just want to do the usual trick of separating the dx pieces from dt pieces in this equation. And we can rewrite that equation as dt is equal to 1 over H naught times x dx over the square root of F. And maybe I'll rewrite it as the square root of F of x to make it explicit that F depends on x. So this is just the rewriting of this equation, moving factors around, and in this form we can integrate it and determine the age of the universe.

The present age of the universe can be obtained just by integrating this expression from the big bang up to the present. And that will be the integral dt from the big bang up to present, the sum of all the time intervals from the big bang until now. And it's just equal to 1 over H zero times the integral of x dx over the square root of F of x. And now-- just to think about the limits of integration-- what should limits of integration be?

AUDIENCE: Zero to one.

PROFESSOR: I hear zero to one, and that's correct. We're integrating from the big bang up to the present. At the big bang, a is equal to zero and therefore x is equal to zero. And, at the present time, t is equal to t naught and therefore x is equal to 1. So we integrate up to one if we want the present age of the universe. We could also integrate it up to any other value of x that we want, and it will tell us the age of universe when the scale factor had that value.

So this is the final, state of the art formula for the age of the universe, expressed in terms of the matter contribution to omega, the radiation contribution to omega, and the vacuum contribution to omega, and the value of H naught. Those are the only ingredients on the right-hand side there. And then you can calculate the age. And it's the completely state of the art formula. It's exactly what the Planck people did when they told you that the age of the universe was 13.9 billion years, using that formula.

Now as far as actually doing the integral, in the general case, the only way to do it is numerically. That's how it's usually done. Special cases can be done analytically. We've already talked about the case where there's no cosmological constants, no vacuum energy, but just matter and curvature-- omega, in this language. There's another special case which can be done, which is the case that involves vacuum energy and nonrelativistic matter. And this is the flat case, only, that can be done analytically.

So it corresponds to omega radiation equals omega k equals zero, and that means that omega matter plus omega vac is equal to 1, because the sum of all the omegas is always equal to 1. So in this case I can write an answer for you. And I don't intend to try to derive this answer, but it's worth knowing that can be written analytically. That's the main point, I guess. So it does get divided into three cases depending on whether omega matter is larger than, smaller than, or equal to 1.

So the first case will be if omega matter zero is greater than 1. And if omega matter is greater than 1, that corresponds to omega vac less than 1 because the sum of the two is equal to 1 in all cases, here. So omega vac has to be less than 0. So this is not our real universe but it's a calculation that you can do, and it's 2 over 3 H naught times the inverse tangent of the square root of omega sub m zero 1 over the square root of omega sub m zero minus 1. So if you plug this integral into mathematica, you should get that answer or something equivalent to it.

For the case-- the borderline case, here-- where omega matter zero equals 1, that's the special case where omega vac is equal to 0 because the sum of these is always one. So this special case in the middle is the case we already knew, it's just the matter-dominated flat universe. So that's two thirds H inverse. So it's 2 over 3 H naught.

And then, finally, if omega matter zero is less than 1 and then omega vac is positive. And this [INAUDIBLE] approximation is our universe, that is, that our universe has possibly zero curvature-- in any case, unmeasureably small curvature-- and very, very small radiation for most of its evolution. So this last case is our universe except for cases that are near the radiation-dominated era, and the formula here is 2 over 3 H naught times the inverse hyperbolic tangent of 1 minus-- square root, excuse me-- of 1 minus omega sub m zero over the square root of 1 minus omega sub m zero.

OK, so this is just a result obtained by doing that integral for the special case that we're talking about. Now, I don't know any simpler way to write it except as these three cases. It is, however, a single analytic function, and when you graph it-- and I'll show you a graph-- it is one smooth function right across the range of these three cases, which is similar to what we found the earlier for the flat, matter-dominated case.

So let's see. This is not that yet. This is the case that we did a long time ago, actually, the case of a matter-dominated universe with nothing but nonrelativistic matter and possibly with curvature. And I can remind you, here, that what we found for that model is that we tended to get ages there were too young. So if we take a reasonable value for H of 67 to 70 kilometers per second per megaparsec-- which is in this range-- and take a reasonable value for omega-- which is somewhere between, say, 0.2 and 1 depending on what you consider reasonable, this model doesn't work anyway-- but if you take omega anywhere between 0.1 and 1, you get numbers for the age which are in the order of 10 billion years, which is not old enough to be consistent with what we know about the ages of the older stars. And especially if you think that omega should be one, you get a very young age of more like 9 billion years, which is what we found earlier. This is just a graph of those same equations.

But, if we include vacuum energy, it makes all the difference. So this now is a graph of those equations. What's shown is the age, T naught, as a function of H and for various values of omega sub m, the same omega sub m that's called omega sub m zero on the blackboard. And shown here are the Barbara Ryden a benchmark point, which is the left-most of these two almost overlapping points. And also shown here is the favored point from the WMAP satellite seven-year data. They lie almost on top of each other.

I didn't get a chance to plot the Planck point, which is the one that we would consider the most authoritative these days, but I'll add that before I post the lectures. It lies almost on top of these, and it corresponds to a Hubble expansion rate of a little under 70, and a vacuum energy contribution of about 0.7, and therefore a matter contribution of about 0.3. This curve.

And it gives an age of 13.7, 13.8 billion years-- perfectly consistent with estimates of the age of the oldest stars. So this age problem which had been, until the discovery of the dark energy, a serious problem in cosmology for many, many years goes away completely once one adds in the dark energy. So that's it for the age calculation. Are there any questions about the age of the universe? Yes?

AUDIENCE: So when you say dark energy, are you using that synonymously with vacuum energy?

PROFESSOR: Sorry, yes. I used the word dark energy there and I've been talking about vacuum energy, and what's the relationship? When I said dark energy I really meant vacuum energy. In general, the way these words are used is that vacuum energy has a very specific meaning. It really does mean the energy of the vacuum, and by definition, therefore, it does not change with time, period. We don't know for sure what this stuff is that's driving the acceleration of the universe, and hence the name dark energy, which is more ambiguous. I think the technical definition of dark energy is it's whatever the stuff is that's driving the acceleration of the universe. And the other conceivable possibility-- and observers are hard at work trying to distinguish, experimentally, between these two options-- the other possibility is that it could be a very slowly evolving scalar field of the same type that drives inflation that we'll be talking about later. But this would be a much lower energy scale than the inflation of the early universe, and much more slowly evolving.

So far, we have not yet found any time variation in the dark energy. So, so far, everything we' have learned about the dark energy is consistent with the possibility that it is simply vacuum energy. Question.

AUDIENCE: Is the amount of dark energy related to the amount of dark matter?

PROFESSOR: Is the amount of dark energy related to. the amount of dark matter? No. They're both numbers and they differ by a factor of 2 and 1/2 or so, but there's no particular relationship between them that we know of.

AUDIENCE: But doesn't dark matter imply that they have a certain attraction to bodies around it, which is a form of energy?

PROFESSOR: Yeah, well let's talk about this later.

AUDIENCE: Do we have any idea what dark energy is at all?

PROFESSOR: OK, the question is, do we have any idea what dark energy is at all? And the answer is probably yes. That is, I think there's a good chance it is vacuum energy. Now if you ask what is vacuum energy, what is it about the vacuum that gives it this nonzero energy, there we're pretty much clueless. I was going to talk about that a little more at the end of today, if we get there. But whatever property of the vacuum it is that gives it its energy-- we know of many, it's just a matter of what dominates and how they add up-- the end result is pretty much the same as far as the phenomenology of vacuum energy. So we understand the phenomenology of vacuum energy, I would say, completely. The big issue, which I'll talk about either at the end of today or next time, is trying to estimate the magnitude of the vacuum energy, and there we're really totally clueless, as I will try to describe.

OK, that's it for my slides. OK, I wanted to now talk about another very important calculation, which is basically the calculation which led to the original evidence that the universe is accelerating to begin with. OK, discovery that the universe was accelerating was made, as I said earlier, by two groups of astronomers in 1998, and the key observation was using a type 1a supernovae as standard candles to measure the expansion rate of the universe versus time, looking back into the past. And basically what they found is that when they look back about 5, 6 billion years, the expansion rate then was actually slower than expansion rate now, meaning that the universe has accelerated. And that was the key observation.

So the question for us to calculate is, what do we expect, as a function of these parameters, for redshift versus luminosity? These astronomers, by using type 1a supernovae as standard candles, are basically using the luminosity measurements of these type 1a supernovae as estimates of their distance. So what they actually measured was simply luminosity verses redshift, and that's what we will learn how to calculate, and the formula that will get will be, again, exactly the formula that they used when they were trying to fit their data-- to understand what their data was telling them-- about possible acceleration of the universe.

So the calculation we're about to do is really nothing new to you folks because we have calculated luminosities in another contexts. Now we will just write down the equations in their full glory, including the contribution due to vacuum energy. So we'd like to do these calculations in a way that allows for curvature, even though-- in the end-- we're going to discover that the curvature of our universe is-- as far as anybody can tell-- negligible. But people still look for it and it still very well could be there at the level of one part in 1,000 or something like that. But at the level of 1 part in 100, it's not there.

So we begin by writing down the Robertson-Walker metric, ds squared is equal to minus c squared dt squared plus a squared of t times dr squared over 1 minus little k times r squared plus r squared d theta squared plus sine squared theta d phi squared, end curly brackets. OK, so this is the metric that we're familiar with. We're going to be interested, mainly, in radial motion, and if you're interested mainly in radial motion, it helps to simplify the radial part of this metric by using a different radial variable. And we've done this before, also.

At this point, we really need to pick whether we're talking about open or closed. If we're talking about flat, we don't need to do anything, really. If you eliminate k, here, the radial part is as simple as it gets. But if we want talk about open or closed, it pays to use different variables, and the variable that we'd use would be different in two cases. So I'm going to consider the closed-universe case.

And I'm going to introduce an angle, sine of psi being equal to the square root of k-- which is positive in this case-- times little r. And this psi is, in fact, if you trace everything back, the angle from the w-axis that we originally used when we constructed the closed Robertson-Walker universe in the first place. But now we're essentially working backwards. We've learned to know and love this expression, so we're going to just rewrite it in terms of the new variable, sin of psi equals the square root of k times r. And from this, by just differentiation, you discover that deep psi is equal to the square root of k times dr over cosine psi. And that is equal to the square root of k times dr over the square root of 1 minus kr squared.

So this, then, fits in very nicely with the metric itself. The metric is just the square of this factor, and therefore it is just proportional to deep psi squared all by itself. And rewriting the whole metric, we can write it as ds squared is equal to minus c squared dt squared plus a new scale factor-- which I'll define in a second in terms of the old one-- times deep psi squared plus-- now, the angular term becomes nonstandard instead of just having an r squared here, we have sine squared of psi. Which is, of course, proportional to r squared. And that multiplies d theta squared plus sine squared theta d phi squared, end curly brackets.

And a tilde is just equal to our original a divided by the square root of k. So we scaled it. And I should mention that I'm putting a tilde here because we've already written an a without a tilde there, and they're not equal to other. If you want to just start here, you can, and then there's no need for the tilde. You could just call this the scale factor and it doesn't need a tilde. The tilde is only to distinguish the two cases from each other.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Sorry?

AUDIENCE: Do a and a tilde have different units?

PROFESSOR: Didn't hear you?

AUDIENCE: Do a and a tilde have different units?

PROFESSOR: They do have different-- yes, a and a tilde do have different units. That's right, and that's because in what one might call conventional units here, r is some kind of a coordinate distance. So in my language I'd measure it in notches, and then a has units of meters per notch. On the other hand, here psi is an angle. It is naturally dimensionless. So one doesn't introduce notches in this case, and therefore a just has units of length-- a tilde, rather-- just has units of length.

OK, now we want to imagine that some distant galaxy is radiating-- or a distant supernova, perhaps-- and we want to ask, what is the intensity of the radiation that we receive on earth? And we'll draw the same picture that we've drawn at least twice before, if not more. We'll put the source in the middle. We'll imagine a sphere surrounding the source, with the source of the center, and we'll imagine that the sphere has been drawn so that our detector is on the surface of the sphere. This will be the detector. And we'll give a symbol for the area of the detector. It will be a. And we'll imagine drawing this in our co-moving coordinate system where psi is our radial variable. So the sphere here will be at some value, psi equals psi sub d-- where d stands for detector-- and psi equals zero at the center.

OK, I'm going to make the same kind of arguments we've made in the past. We say that the fraction of light that hits the sphere-- which hits the detector-- is just equal to the area of the detector over the area of the sphere. Now, the area of detector is, by definition, a. The area of the sphere we have to be a little bit careful about because we have to calculate the area of the sphere using the metric.

Now, the metric is slightly nontrivial, but the sphere is just described by varying theta and phi. And if we just vary theta and phi, this piece of the metric is what we're used to-- it's the standard Euclidean spherical element-- and the coefficient that multiplies is just the square of the radius of that sphere. So the radius of our sphere is a tilde times sine psi. That's the important thing that we get from the metric. The thing that multiplies d theta squared and d phi squared, et cetera, is just the square of the radius of the sphere that determines distances on the surface of the sphere.

So what goes here is 4 pi times the radius squared. So it's 4 pi times a tilde squared of t naught times sine squared of psi sub d. It's t naught because we're interested in what happens when we detect this radiation today. Our detector is detecting it today and has area a today, and we want to compare it with entire sphere that surrounds this distant source as that sphere appears today, so that all of the distances are measured today, and therefore can be properly compared.

The other thing we have to remember is the effect of the redshift. The redshift, we've said earlier, and it's just a repetition, it reduces the energy of each photon by a factor of 1 plus z, the redshift, and similarly it reduces the rate at which photons are arriving at the sphere by that same factor-- 1 plus z. It basically says that any clock slows down by a factor of 1 plus z, and that clock could be the frequency of the photon-- which affects its energy proportionally-- or the arrival rate of the photons. That's also a clock that get time dilated in the same way.

So we get two factors of 1 plus z sub s, I'll call it. s for z of the source, the z between the time of emission at the source to a time where it arrives at us today. So 1 plus z is equal to a of t naught divided by a of t emission. I'll just put it here to remind us. One from redshift of photons and one from arrival rate.

OK, putting that together we can now say that the total power received is equal to the power originally emitted by the source-- p will just be the power emitted by the source-- divided by 1 plus the redshift z of the source squared, and then just times the fraction. A over 4 pi a twiddle squared sine squared psi d. And then, finally, what we're really interested in is J-- the intensity of the source as we measure it-- which is just the power received by our detector divided by its area.

So from this formula we just get rid of the A there. We can write it as the power emitted by the source, capital P, divided by 4 pi 1 plus z sub s squared a twiddle squared of t naught times sine squared psi sub d. Now that effectively is the answer to this question except that we prefer to rewrite it in terms of things that are more directly meaningful to astronomers. a twiddle is not particularly meaningful to the astronomer. The redshift is, that's OK. But a twiddle is not particularly meaningful to an astronomer, nor is sine squared of psi sub d.

Now, many astronomers who know general relativity can figure this out, of course, but it's our job to figure it out. We would like to express this in terms of things that are directly measured by astronomers. So to do that, first of all, a tilde-- to get a tilde related to other things, it really just goes back to the definition that we gave for omega sub k sub 0. And if you look back at that definition, you'll find that a of t naught tilde is just equal to c times the inverse of the present Hubble expansion rate times the square root of minus omega sub k comma zero.

And this is for the close-universe case. The closed-universe case, little k is positive. But if you remember the definition of omega sub k naught-- maybe I should write it back on the blackboard, or is it findable? It's not findable. The original definition big A for this omega sub k naught was just minus Kc squared over a squared of t naught H naught squared. So this is just rewriting of that, and for our closed-universe case, k is positive, omega sub k naught is negative, this is then the square root of a positive number with that extra minus sign, so everything fits together. So that takes care of expressing a tilde in terms of measurable things. We use this formula. Expansion rate is measurable, omega sub k comma zero is measurable.

And then, in terms of sine squared psi sub d, we obtain that by reminding ourselves that we know how to trace light rays through this universe. Light waves just travel locally at the speed c, they travel locally on null geodesics. So if we're looking at a radial light ray, this metric tells us-- if we apply it to a radial light ray where ds squared equals minus dc squared-- where ds squared has to be zero-- that says that minus c squared dt squared plus a twiddle squared of t d psi squared equals zero. This is just the equation that says we have a null line, a null radial line. That implies that deep psi dt has to equals c over a twiddle of t, which is a formula that, in other cases, we try to motivate just by using intuition. But, in that case, we were probably not talking about curved universe where the intuition is a little bit less strong. But you see it does follow immediately from assuming that we're talking about a null geodesic in the Robertson-Walker metric.

Now, the point is that the Friedmann equation, which we've been writing and rewriting, tells us what to do with that. The Friedmann equation basically allows us to integrate that because it allows us to express a in terms of x, and we know some things about x. So let me try to get that on the blackboard, here. We know that H squared-- which is x dot over x squared-- can be written as H zero squared over x to the fourth times this famous function F of x. And psi of a given redshift, according to this equation, could just be obtained by integration of the time that the source emits the radiation up to the present time of c over a twiddle of t times dt.

And now to rewrite this in terms of redshift, we can use the fact that 1 plus z is equal to 1 over x because we know how to relate 1 plus z to the scale factor. 1 plus z is just the ratio of scale factors and it's precisely the ratio that we called 1 over x. And we can then differentiate this equation and find that dz is equal to minus a twiddle of t naught over a twiddle of t squared times a twiddle dot times dt, rewriting x in terms of a of t naught over a of t.

And this, then, is equal to minus a twiddle of t naught times H of t times, oops, times dt over a tilde of t. And this allows us to replace the dt that appears there and the final relationship is that psi of s is equal to 1 over a tilde of t naught times the integral from zero up to z sub s of c over H of z dz. Yeah, I think that looks like it works.

So it really is just a matter of changing variables to express things in terms of H and integrating over z instead of integrating over t. And the usefulness of that is simply that z is the variable that astronomers use to measure time. And this then can be written in more detail, and it really finishes the answer more or less. Psi of z sub s can be written just-- writing in what a tilde is according to our definition here-- square root of the magnitude of omega comma k zero-- this could also have been written as minus omega of k comma 0 because we know it's a negative quantity-- and then times the integral from 0 to z sub s and integral dz.

Now I'm just writing H as a function of z. Earlier we had written H-- it's no longer on the screen, I guess-- earlier we had written H in terms of F of z, oh, excuse me, F of x. x is related to z simply by this formula. So since the integral was written with z as the variable of integration, I'm going to rewrite the integrand in terms of z, but it really is just our old friend F of x. So it would be the square root of omega sub m zero 1 plus z cubed plus omega sub radiation zero times 1 plus z to the fourth, all inside the square root, here, plus omega sub vac zero plus omega sub k zero times 1 plus z squared. And this, then, is the answer for psi of z.

And then we put that into here and replace a twiddle by this, and we get a formula for what we're looking for, an expression for the actual measured intensity of the source at the Earth in terms of the parameters chosen here-- the current values of omega and the redshift of the source. And that's all that goes into this final formula. So if you know the current values of omega and the redshift of the source, you can calculate what you expect the measure intensity to be in terms of the intrinsic intensity. And that's exactly what the supernova people did in 1998 using exactly this formula-- nothing different-- and discovered that, in order to fit their data, they needed a very significant contribution from this vacuum energy, namely a contribution in the order of 60 or 70%.

So we will stop there for today. We will continue on Thursday to talk a little bit more about the physics of vacuum energy.