MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 20, 2013
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# PROBLEM SET 8

**DUE DATE:** Friday, November 22, 2013. FYI, Problem Set 9 will be due just after Thanksgiving, on Monday, December 2, and it will be the last problem set before Quiz 3. There will also be a Problem Set 10, to be due Tuesday, December 10, 2013.

**READING ASSIGNMENT:** Barbara Ryden, *Introduction to Cosmology,* Chapter 9 (*The Cosmic Microwave Background*).

**UPCOMING QUIZ:** Thursday, December 5, 2013.

## PROBLEM 1: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT *(25 points)*

In Lecture Notes 7, we derived the relation between the power output $P$ of a source and the energy flux $J$, for the case of a closed universe:

$$J = \frac{P H_0^2 |\Omega_{k,0}|}{4\pi(1+z_S)^2 c^2 \sin^2 \psi_D} \ ,$$

where

$$\psi_D = \sqrt{|\Omega_{k,0}|} \int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z)^2}} \ .$$

Here $z_S$ denotes the observed redshift, $H_0$ denotes the present value of the Hubble constant, $\Omega_{m,0}$, $\Omega_{\rm rad,0}$, and $\Omega_{\rm vac,0}$ denote the present contributions to $\Omega$ from nonrelativistic matter, radiation, and vacuum energy, respectively, and $\Omega_{k,0} \equiv 1 - \Omega_{m,0} - \Omega_{\rm rad,0} - \Omega_{\rm vac,0}$.

(a) Derive the corresponding formula for the case of an open universe. You can of course follow the same logic as the derivation in the lecture notes, but the solution you write should be complete and self-contained. (I.e., you should **NOT** say "the derivation is the same as the lecture notes except for . . . .")

(b) Derive the corresponding formula for the case of a flat universe. Here there is of course no need to repeat anything that you have already done in part (a). If you wish you can start with the answer for an open or closed universe, taking the limit as $k \to 0$. The limit is delicate, however, because both the numerator and denominator of the equation for $J$ vanish as $\Omega_{k,0} \to 0$.

**PROBLEM 2: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF** *(20 points)*

**READ THIS:** *This problem was Problem 8 of Review Problems for Quiz 3 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqr3-1.pdf. Like Problem 4 of Problem Set 3 and Problem 3 of Problem Set 6, but unlike all other homework problems so far, in this case you are encouraged to look at the solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in Problem 7 of Review Problems for Quiz 3, from 2011. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where $V$ is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/a^{3/2}(t)$.

Suppose that you are given the present value of the Hubble parameter $H_0$, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\mathrm{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe $t_0$ in terms of these quantities.

(a) *(10 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{a(t)}{a(t_0)}$$

for an arbitrary time $t$. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(10 points)* Write an integral expression for the age of the universe $t_0$. The expression should depend only on $H_0$ and the various contributions to $\Omega_0$ listed above ($\Omega_{m,0}$ , $\Omega_{r,0}$ , etc.), but it might include $x$ as a variable of integration.

**PROBLEM 3: SHARED CAUSAL PAST** *(20 points)*

Recently several of my colleagues published a paper (Andrew S. Friedman, David I. Kaiser, and Jason Gallicchio, "The Shared Causal Pasts and Futures of Cosmological Events," http://arxiv.org/abs/arXiv:1305.3943, *Physical Review D,* Vol. 88, article 044038 (2013)) in which they investigated the causal connections in the standard cosmological model. In particular, they calculated the present redshift $z$ of a distant quasar which has the property that a light signal, if sent from our own location at the instant of the big bang, would have just enough time to reach the quasar and return to us, so that we could see the reflection of the signal at the present time. They found $z = 3.65$, using $\Omega_{\mathrm{vac},0} = 0.685$, $\Omega_{\mathrm{matter},0} = 0.315$, $\Omega_{\mathrm{rad},0} = 0$, and $H_0 = 67.3$ km-s$^{-1}$-Mpc$^{-1}$. Feel free to read their paper if you like. Your job, however, is to carry out an independent calculation to find out if they got it right. You will encounter an integral which you may decide to evaluate numerically.

**PROBLEM 4: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT — NUMERICAL INTEGRATION** *(EXTRA CREDIT, 20 pts)*

Calculate numerically the result from Problem 4 for the case of a flat universe in which the critical density is comprised of nonrelativistic matter and vacuum energy (cosmological constant). Specifically, calculate numerical values for $J/(PH_0^2)$ as a function of $z$, for $\Omega_{m,0} = 0.3$ and $\Omega_{\text{vac},0} = 0.7$. Compute a table of values for $z = 0.1, 0.2, 0.3, \ldots, 1.5$. Feel free to attach a computer printout of these results, but be sure that it is labeled well enough to be readable to someone other than yourself. (If you are not confident in the expression that you obtained in Problem 4 for the flat universe case, you can for equal credit do this problem for an open universe, with $\Omega_{m,0} = 0.3$ and $\Omega_{\text{vac},0} = 0.6$.) For pedagogical purposes you are asked to compute these numbers to 5 significant figures, although one does not need nearly so much accuracy to compare with data. For the fun of it, the solutions will be written to 15 significant figures. Note that the speed of light is now *defined* to be 299,792,458 m/s.

**PROBLEM 5: PLOTTING THE SUPERNOVA DATA** *(EXTRA CREDIT, 20 pts)*

The original data on the Hubble diagram based on Type Ia supernovae are found in two papers. One paper is authored by the High Z Supernova Search Team,* and the other is by the Supernova Cosmology Project.† More recent data from the High Z team, which includes many more data points, can be found in Riess *et al.*, http://arXiv.org/abs/astro-ph/0402512.¶ (By the way, the lead author Adam Riess was an MIT undergraduate physics major, graduating in 1992.)

You are asked to plot the data from either the 2nd or 3rd of these papers, and to include on the graph the theoretical predictions for several cosmological models.

The plot will be similar to the plots contained in these papers, and to the plot on p. 121 of Ryden's book, showing a graph of (corrected) magnitude $m$ vs. redshift $z$. Your graph should include the error bars. If you plot the Perlmutter *et al.* data, you will be plotting "effective magnitude" $m$ vs. redshift $z$. The magnitude is related to the flux $J$ of the observed radiation by $m = -\frac{5}{2}\log_{10}(J) + \text{const}$. The value of the constant in this expression will not be needed. The word "corrected" refers both to corrections related to the spectral sensitivity of the detectors and to the brightness

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* http://arXiv.org/abs/astro-ph/9805201, later published as Riess *et al.*, *Astronomical Journal* **116**, 1009 (1998).

† http://arXiv.org/abs/astro-ph/9812133, later published as Perlmutter *et al.*, *Astrophysical Journal* **517**:565–586 (1999).

¶ Published as *Astrophysical Journal* **607**:665-687 (2004).

of the supernova explosions themselves. That is, the supernova at various distances are observed with different redshifts, and hence one must apply corrections if the detectors used to measure the radiation do not have the same sensitivity at all wavelengths. In addition, to improve the uniformity of the supernova as standard candles, the astronomers apply a correction based on the duration of the light output. Note that our ignorance of the absolute brightness of the supernova, of the precise value of the Hubble constant, and of the constant that appears in the definition of magnitude all combine to give an unknown but constant contribution to the predicted magnitudes. The consequence is that you will be able to move your predicted curves up or down (i.e., translate them by a fixed distance along the $m$ axis). You should choose the vertical positioning of your curve to optimize your fit, either by eyeball or by some more systematic method.

If you choose to plot the data from the 3rd paper, Riess *et al.* 2004, then you should see the note at the end of this problem.

For your convenience, the magnitudes and redshifts for the Supernova Cosmology Project paper, from Tables 1 and 2, are summarized in a text file on the 8.286 web page. The data from Table 5 of the Riess *et al.* 2004 paper, mentioned above, is also posted on the 8.286 web page.

For the cosmological models to plot, you should include:

(i) A matter-dominated universe with $\Omega_m = 1$.

(ii) An open universe, with $\Omega_{m,0} = 0.3$.

(iii) A universe with $\Omega_{m,0} = 0.3$ and a cosmological constant, with $\Omega_{\text{vac},0} = 0.7$. (If you prefer to avoid the flat case, you can use $\Omega_{\text{vac},0} = 0.6$. Or, if you want to compare directly with Figure 4 of the Riess *et al.* (2004) paper, you should use $\Omega_{m,0} = 0.29$, $\Omega_{\text{vac},0} = 0.71$.)

You may include any other models if they interest you. You can draw the plot with either a linear or a logarithmic scale in $z$. I would recommend extending your theoretical plot to $z = 3$, if you do it logarithmically, or $z = 2$ if you do it linearly, even though the data does not go out that far. That way you can see what possible knowledge can be gained by data at higher redshift.

**NOTE FOR THOSE PLOTTING DATA FROM RIESS ET AL. 2004:**

Unlike the Perlmutter *et al.* data, the Riess *et al.* data is expressed in terms of the distance modulus, which is a direct measure of the luminosity distance. The distance modulus is defined both in the Riess *et al.* paper and in Ryden's book (p. 120) as

$$\mu = 5 \log_{10} \left( \frac{d_L}{1 \text{ Mpc}} \right) + 25 ,$$

where Ryden uses the notation $m - M$ for the distance modulus, and $d_L$ is the luminosity distance. The luminosity distance, in turn, is really a measure of the observed brightness of the object. It is defined as the distance that the object would have to be located to result in the observed brightness, if we were living in a static Euclidean universe. More explicitly, if we lived in a static Euclidean universe and an object radiated power $P$ in a spherically symmetric pattern, then the energy flux $J$ at a distance $d$ would be

$$J = \frac{P}{4\pi d^2} \ .$$

That is, the power would be distributed uniformly over the surface of a sphere at radius $d$. The luminosity distance is therefore defined as

$$d_L = \sqrt{\frac{P}{4\pi J}} \ .$$

Thus, a specified value of the distance modulus $\mu$ implies a definite value of the ratio $J/P$.

In plotting a theoretical curve, you will need to choose a value for $H_0$. Riess *et al.* do not specify what value they used, but I found that their curve is most closely reproduced if I choose $H_0 = 66$ km-sec$^{-1}$-Mpc$^{-1}$. This seems a little on the low side, since the value is usually estimated as 70–72 km-sec$^{-1}$-Mpc$^{-1}$, but Riess *et al.* emphasize that they were not concerned with this value. They were concerned with the relative values of the distance moduli, and hence the shape of the graph of the distance modulus vs. $z$. In their own words, from Appendix A, "The zeropoint, distance scale, absolute magnitude of the fiducial SN Ia or Hubble constant derived from Table 5 are all closely related (or even equivalent) quantities which were arbitrarily set for the sample presented here. Their correct value is not relevant for the analyses presented which only make use of differences between SN Ia magnitudes. Thus the analysis are independent of the aforementioned normalization parameters."

**Total points for Problem Set 8: 65, plus an optional 40 points of extra credit.**