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Computational Problem Set 2
Polytropic Models for Stars

Optional

1. Integrating the Lane-Emden Equation

Integrate the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n$$

for polytropic indices of $n = 1.0, 1.5, 2.0, 2.5, 3.0$, and 3.5.

Break up this second order differential equation into two first-order, coupled equations in $d\phi/d\xi \equiv u$ and $du/d\xi$. Use a 4th-order Runge-Kutta integration scheme or some other equivalent integration method to find $\phi(\xi)$. Recall the boundary conditions at the center:

$$u(0) = 0 \quad \text{and} \quad \phi(0) = 1 \quad .$$

Use the analytic expansion for $\phi(\xi)$ near the center:

$$\phi(\xi) = 1 - \frac{\xi^2}{6}$$

to help start the integration. The surface of the star is defined by $\phi(\xi_1) = 0$.

Plot the dimensionless temperature, $\phi(\xi)$, and the dimensionless density, $\phi^n(\xi)$, for all 6 values of n. It would be best to put all the temperature plots on one graph and all the density plots on another.

2. Tabulating Some Physical Properties of Polytropes

As the integrations in part (1) are underway, compute for each model the dimensionless potential energy, Ω (in units of $-GM^2/R$), and the dimensionless moment of inertia, k (in units of MR^2). Tabulate ξ_1, $-(d\phi/d\xi)_{\xi_1}$, Ω, and k for each of the 6 polytropic models.

3. Model of the Sun

Use an n = 3 polytropic model to represent the internal structure of the Sun. The two parameters to fix are $M = M_\odot$ and $R = R_\odot$.

(a) Plot the physical temperature (in K) vs. radial distance in units of $r/R_\odot$. Plot $\log T$ vs. $r/R_\odot$. Do the same for the physical density (g cm^{-3}). Again, plot $\log\rho$ vs. $r/R_\odot$. Instead of using the values for the central density, ρ_0, and central temperature, T_0, deduced for an $n = 3$ polytrope with $M = M_\odot$ and $R = R_\odot$, take the "known" values of $\rho_0 = 158$ g cm^{-3} and $T_0 = 15.7 \times 10^6$ K.

(b) Compute the nuclear luminosity of the sun using the above temperature and density profiles. Take the nuclear energy generation rate to be:

$$\epsilon(\rho, T) = 2.46 \times 10^6 \rho^2 X^2 T_6^{-2/3} exp(-33.81 T_6^{-1/3}) \quad \text{ergs cm}^{-3} \text{ sec}^{-1}$$

where ρ is in g cm^{-3}, T_6 is the temperature in units of 10^6 K, and X is the hydrogen mass fraction (take $X = 0.6$). Note that the units of $\epsilon(\rho, T)$ are ergs cm^{-3} sec^{-1} and not ergs g^{-1} sec^{-1} as given in the lecture. Reduce the problem to a dimensioned constant times an integral involving only functions of ϕ and ξ. (There will also appear a "T_0" inside the integral for which you can plug in the value of 15.7×10^6 K, or $T_{06} = 15.7$.) Show the value of your constant and the form of the dimensionless integral. Evaluate the nuclear luminosity of the Sun in units of ergs sec^{-1}.