8.251 – Homework 8

B. Zwiebach Spring 2007

Due Thursday, April 12.

1. (15 points) *Closed strings in the light-cone gauge.*

   Consider a closed string for which

   $$X^{(2)}(\tau,\sigma) = \sqrt{2\alpha'}\Big(a\sin(\tau-\sigma) + b\cos(\tau-\sigma) + \bar{a}\sin(\tau+\sigma) + \bar{b}\cos(\tau+\sigma)\Big)\,,$$

   and all other transverse coordinates $X^I(\tau,\sigma)$ vanish. In the above, $a, b, \bar{a}$, and $\bar{b}$ are real constants. Note that a and b are the coefficients of waves that propagate towards increasing σ while $\bar{a}$ and $\bar{b}$ are the coefficients of waves that propagate towards decreasing σ. As usual for closed strings, we set $X^+(\tau,\sigma) = \alpha'\, p^+\tau$ and use $\sigma \in [0, 2\pi]$.

   (a) As it turns out, it is not possible to generate a solution of the equations of motion for completely arbitrary values of the constants $a, b, \bar{a}$, and $\bar{b}$. Examine the calculation of $X^-(\tau,\sigma)$ and derive the constraint that the constants must satisfy.

   (b) Your constraint must allow $a = b = \bar{a} = \bar{b} = r$. Use these values to calculate $X^-(\tau,\sigma)$ and to determine the mass of this string.

2. (10 points) Problem 10.2.

3. (10 points) Problem 10.5.

4. (15 points) Problem 10.6.

5. (20 points) Problem 10.7. A slightly improved version of the problem is reproduced here!

   The purpose of this problem is to understand the massive version of Maxwell fields. We will see that in D-dimensional spacetime a massive vector field has $(D-1)$ degrees of freedom.

   Consider the action $S = \int d^Dx\,\mathcal{L}$ with

   $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + mA^\mu\,\partial_\mu\phi\,.$$

   The first term in $\mathcal{L}$ is the familiar one for the Maxwell field. The second looks like a mass term for the Maxwell field, but alone would not suffice. The extra terms show a real scalar field ϕ that, as we shall see, is *eaten* to give the gauge field a mass.

   (a) Show that the Lagrangian $\mathcal{L}$ is invariant under the infinitesimal gauge transformation

   $$\delta A_\mu = \partial_\mu\epsilon\,, \quad \delta\phi = \ldots\,,$$

   where the dots denote an expression that you must determine. While the gauge field has the familiar Maxwell gauge transformation, it is unusual to have a real scalar field with a gauge transformation.

(b) Vary the action and write down the field equations for A^μ and for ϕ.

(c) Argue that the gauge transformations allow us to set $\phi = 0$. Since the field ϕ disappears from sight, we say it is *eaten*. What do the field equations in part (b) simplify into?

(d) Write the simplified equations in momentum space and show that for $p^2 \neq -m^2$ there are no nontrivial solutions, while for $p^2 = -m^2$ the solution implies that there are $D-1$ degrees of freedom. (It may be useful to use a Lorentz transformation to represent the vector p^μ which satisfies $p^2 = -m^2$ as a vector that has a component only in one direction.)