8.07 Lecture 37: December 12, 2012 (THE LAST!)

RADIATION

Radiation: Electromagnetic fields that carry energy off to infinity.

At large distances, $\vec{E}$ and $\vec{B}$ fall off only as $1/r$, so the Poynting vector falls off as $1/r^2$. If the Poynting vector is then integrated over a large sphere, of area $4\pi r^2$, the contribution approaches a constant as $r \to \infty$.

Recall the Liénard-Wiechert potentials:

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_p|(1-\frac{\vec{v}_p}{c}\cdot\hat{\boldsymbol{\mathscr{r}}})}} \ , \tag{1}$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{q\vec{v}_p}{|\vec{r}-\vec{r}_p|(1-\frac{\vec{v}_p}{c}\cdot\hat{\boldsymbol{\mathscr{r}}})} = \frac{\vec{v}_p}{c^2}V(\vec{r},t)} \ , \tag{2}$$

where $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle at t_r,

$$t_r = t - \frac{|\vec{r}-\vec{r}_p|}{c} \ , \tag{3}$$

and

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|} \ , \qquad \boldsymbol{\mathscr{r}} = |\vec{r}-\vec{r}_p| \ . \tag{4}$$

Electric Dipole Radiation

Simplest dipole: two tiny metal spheres separated by a distance d along the z-axis, connected by a wire, with charges

$$q(t) = q_0 \cos(\omega t) \qquad (5)$$

on the top sphere, and $q(t) = -q_0 \cos(\omega t)$ on the bottom sphere.

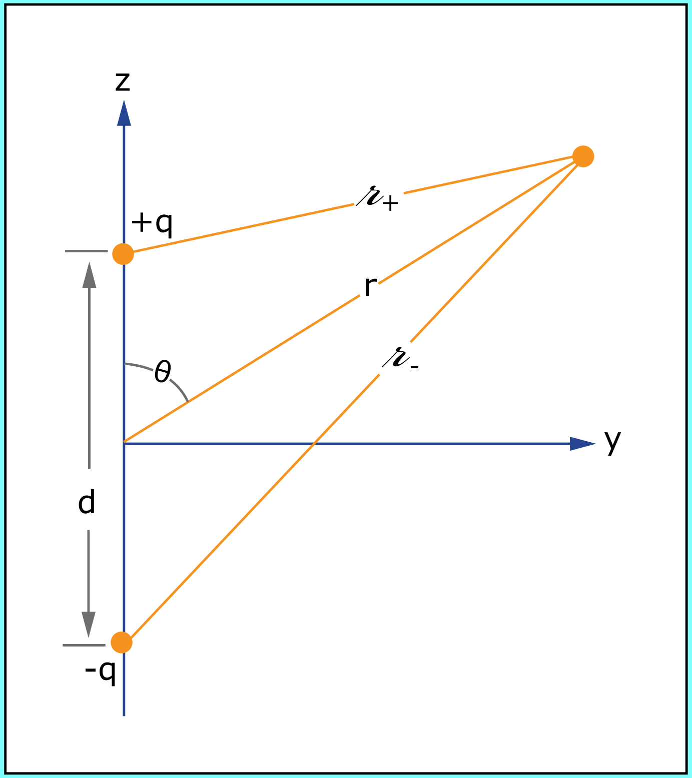

Image by MIT OpenCourseWare.

Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0 \cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0 \cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\} \qquad (6)$$

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Approximation 1: $d \ll r$.

$$\mathscr{r}_{\pm} = \sqrt{r^2 \mp r d \cos\theta + (d/2)^2}$$

$$\Longrightarrow \quad \begin{cases} \frac{1}{\mathscr{r}_{\pm}} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \\ \cos[\omega(t - \mathscr{r}_{\pm}/c)] \simeq \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right] \end{cases} \tag{7}$$

$d \ll r$ is ALWAYS valid for radiation, which is defined in the $r \to \infty$ limit.

Approximation 2: $d \ll \frac{c}{\omega}$.

Since $\lambda = 2 \;\; c/\omega$, this is equivalent to $d \ll \lambda$. This is the IDEAL DIPOLE APPROXIMATION. This is really the first term in a power expansion in d/λ, the multipole expansion for radiation, but we will go no further than the dipole. Implies

$$\cos[\omega(t - \mathscr{r}_{\pm}/c)] \simeq \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta \sin[\omega(t - r/c)] \tag{8}$$

Then, defining $p_0 = q_0\, d$,

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r}\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\} . \quad (9)$$

Approximation 3: $r \gg \lambda$.

The region $r \gg \lambda$ is called the *radiation zone.* This approximation is ALWAYS valid for discussing radiation. Implies that the first term in curly brackets can be dropped:

$$\boxed{V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)] \ .} \quad (10)$$

Summary of approximations: $d \ll \lambda \ll r$.

Need also $\vec{A}$, which is due to the current in the wire. Recall

$$\vec{A}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \, \frac{\vec{J}(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \; . \tag{11}$$

Here

$$\mathrm{d}^3x' \, \vec{J} = I \mathrm{d}\vec{\ell} = \frac{\mathrm{d}q}{\mathrm{d}t} \, \mathrm{d}z \, \hat{z} \tag{12}$$

so

$$\begin{aligned} \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} \, \mathrm{d}z \\ &= \boxed{-\frac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t - r/c)] \, \hat{z}} \end{aligned} \tag{13}$$

Differentiating,

$$\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r} \right) \cos[\omega(t - r/c)]\,\hat{\theta}\ .} \tag{14}$$

$$\boxed{\vec{B}(\vec{r}, t) = \frac{1}{c}\,\hat{r} \times \vec{E}(\vec{r}, t)\ .} \tag{15}$$

Poynting Vector:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{r}\ . \tag{16}$$

Intensity:

Average the Poynting vector over a complete cycle: $\langle \cos^2 \rangle = 1/2$

$$\left\langle \vec{S} \right\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{r}\ . \tag{17}$$

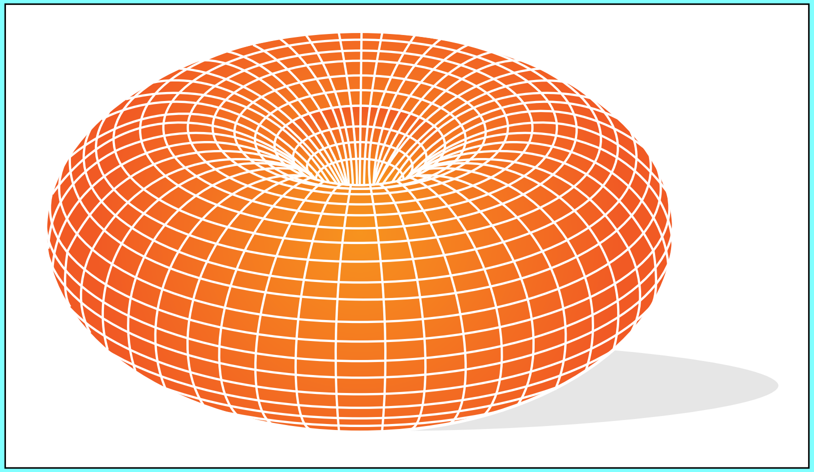

Image by MIT OpenCourseWare.

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Polarized Blue Skies

- Blue is the highest frequency in the visible spectrum. The sky appears blue largely because the ω^4 factor in dipole radiation implies that blue light is scattered more strongly than other frequencies.
- Sunsets are red because the blue light has been scattered out of the path of the sunlight.

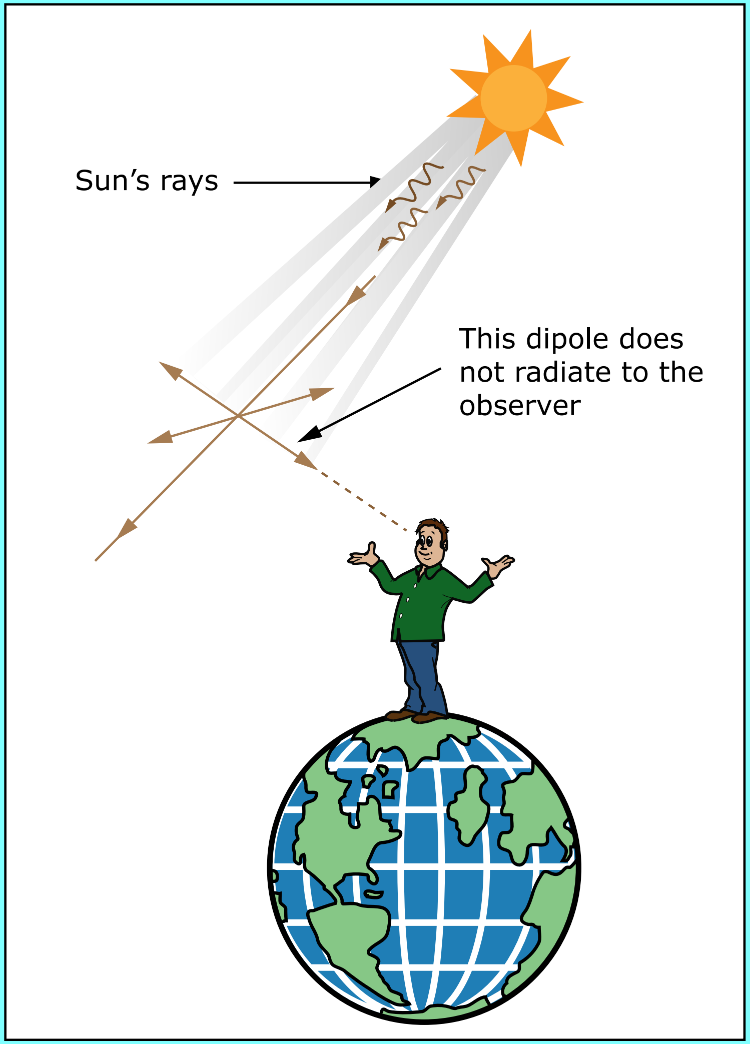

Image by MIT OpenCourseWare.

★ When the line of sight is perpendicular to the Sun's rays, the light is polarized horizontally.

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Total Power:

Integrate over a sphere at large r.

$$\langle P \rangle = \int \left\langle \vec{S} \right\rangle \cdot \mathrm{d}\vec{a} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \int \frac{\sin^2\theta}{r^2}\, r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$$
$$= \boxed{\frac{\mu_0 p_0^2 \omega^4}{12\pi c}\ .} \qquad (18)$$

Magnetic Dipole Radiation

Consider a wire loop of radius b, with alternating current

$$I(t) = I_0 \cos(\omega t) \ , \qquad (19)$$

with magnetic dipole moment

$$\begin{aligned} \vec{m}(t) &= \pi b^2 I(t)\,\hat{z} \\ &= m_0 \cos(\omega t)\,\hat{z} \ , \end{aligned} \qquad (20)$$

where

$$m_0 \equiv \pi b^2 I_0 \ . \qquad (21)$$

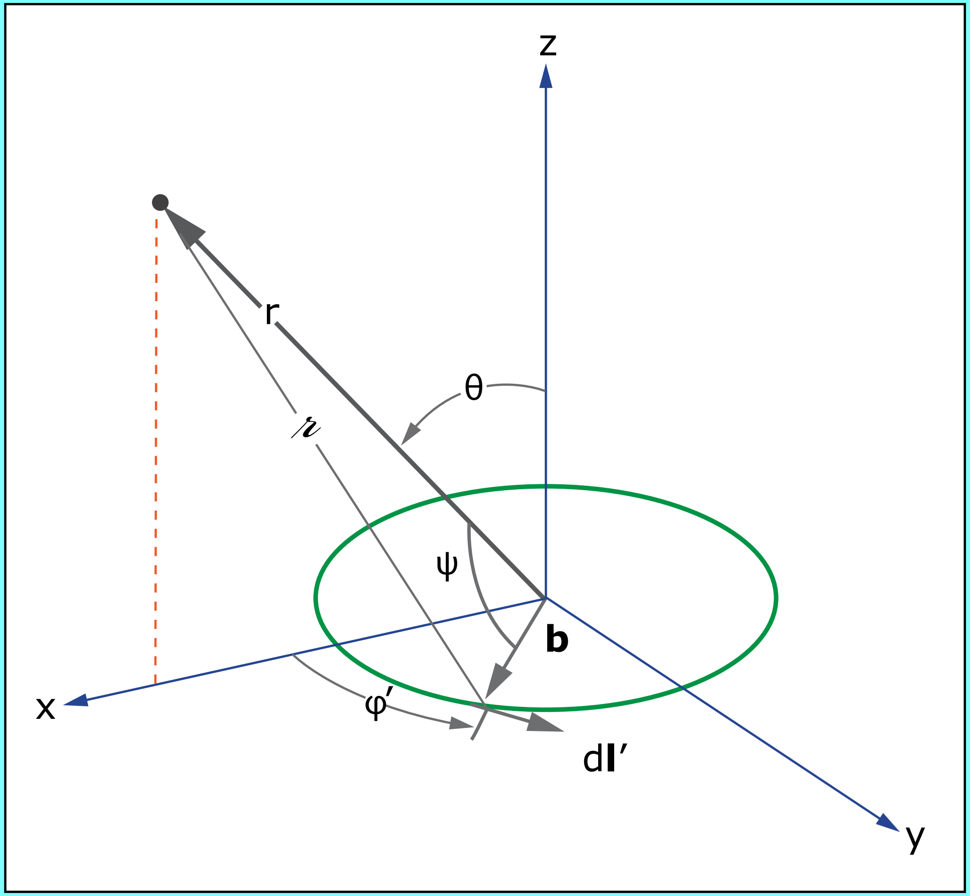

Image by MIT OpenCourseWare.

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Make the same approximations as before, with b replacing d as the size of the system: $b \ll \lambda \ll r$. Find

$$\boxed{\vec{E} = -\frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\phi}\ .} \tag{22}$$

Compared to the electric dipole radiation,

$$p_0 \to \frac{m_0}{c}\ , \qquad \hat{\theta} \to -\hat{\phi}\ . \tag{23}$$

As always for radiation,

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{r} \times \vec{E}(\vec{r},t)\ .} \tag{24}$$

(Electric Dipole) Radiation From an Arbitrary Source

Consider an arbitrary time-dependent charge distribution $\rho(\vec{r}, t)$. Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \, d\tau' \; . \qquad (25)$$

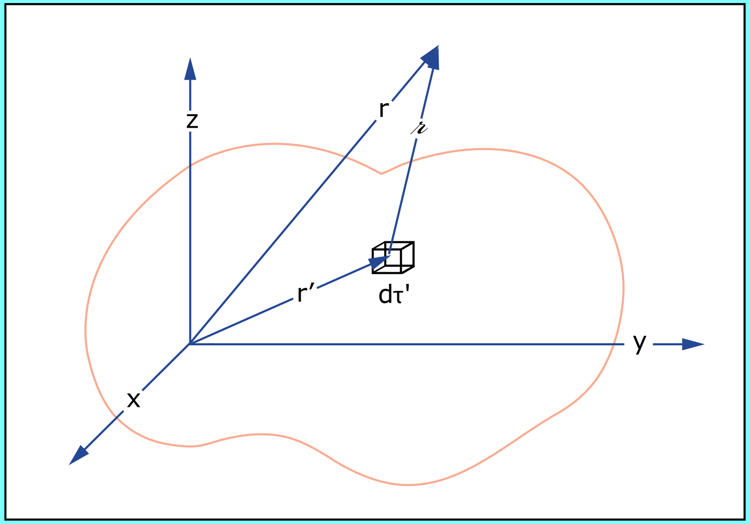

Image by MIT OpenCourseWare.

Expand $1/|\vec{r} - \vec{r}\,'|$ and t_r in powers of $\vec{r}\,'$, using similar approximations as before. Minor difference: here we have no ω. Previously we assumed that $d \ll \lambda$ or equivalently $\omega \ll c/d$. Here we need to assume that $|\ddot{\rho}/\dot{\rho}| \ll c/d$, with similar bounds on higher time derivatives.

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

We find

$$V(\vec{r},t) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc}\right] \;, \tag{26}$$

where t_0 is the retarded time at the origin. Final result:

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &\simeq \frac{\mu_0}{4\pi r}[(\hat{r}\cdot\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}] \\ \vec{B}(\vec{r},t) &\simeq -\frac{\mu_0}{4\pi rc}[\hat{r}\times\ddot{\vec{p}}] \;. \end{aligned}} \tag{27}$$

Although this looks different, it is really the same as what we had for the simple electric dipole, changing to vector notation and replacing $-\omega^2\vec{p}_0$ by $\ddot{\vec{p}}$.

The electric dipole radiation formula is really the first term in a doubly infinite series. There is electric dipole, quadrupole, ... radiation, and also magnetic dipole, quadrupole, ... radiation.

Radiation by Point Charges

Recall the fields (found by differentiating the Liénard-Wiechert potentials):

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{|\vec{r}-\vec{r}_p|}{(\vec{u}\cdot(\vec{r}-\vec{r}_p))^3}\left[(c^2-v_p^2)\vec{u}+(\vec{r}-\vec{r}_p)\times(\vec{u}\times\vec{a}_p)\right] \;, \tag{28}$$

where

$$\vec{u} = c\,\hat{\boldsymbol{\mathscr{r}}} - \vec{v}_p \;, \qquad \vec{\boldsymbol{\mathscr{r}}} = \vec{r} - \vec{r}_p \;. \tag{29}$$

If $\vec{v}_p = 0$ (at the retarded time), then

$$\vec{E}_{\rm rad} = \frac{q}{4\pi\epsilon_0 c^2|\vec{r}-\vec{r}\,'|}\left[\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\vec{a}_p)\right] . \tag{30}$$

Poynting Vector (particle at rest):

$$\vec{S}_{\rm rad} = \frac{1}{\mu_0 c} |\vec{E}_{\rm rad}|^2 \, \hat{\boldsymbol{r}} = \boxed{\frac{\mu_0 q^2 a^2}{16\pi^2 c} \left(\frac{\sin^2\theta}{\boldsymbol{r}^2}\right) \hat{\boldsymbol{r}} \ .} \tag{31}$$

Total Power (Larmor formula):

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c} \ .} \tag{32}$$

Total Power (Larmor formula):

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}\ .} \tag{32}$$

Liénard's Generalization if $\vec{v}_p \neq 0$:

$$\boxed{P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right) = \underbrace{\frac{\mu_0 q^2}{6\pi m_0^2 c}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}}_{\text{For relativists only}}\ .} \tag{33}$$

$P =$ rate at which energy that is destined to become leaving the particle.

Radiation Reaction

If an accelerating particle radiates energy, it must lose energy — obviously! But it is not clear exactly how. Point charges and conservation laws cannot be combined rigorously.

For a nonrelativistic particle, radiation power is given by the Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \ .$$

Maybe then,

$$\vec{F}_{\text{rad}} \cdot \vec{v} \stackrel{?}{=} -\frac{\mu_0 q^2 a^2}{6\pi c} \ . \tag{34}$$

But not necessarily, since P_{Larmor} represents the power destined to become radiation. There can be other energy exchanges with the near fields. Since the RHS of Eq. (34) does not depend on $\vec{v}$, there is no way to match the two sides.

For cyclic motion, the total energy loss over one cycle should match the energy loss described by the Larmor formula. If a cycle extends from t_1 to t_2 then

$$\int_{t_1}^{t_2} a^2\,\mathrm{d}t = \int \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\,\mathrm{d}t = \underbrace{\left.\left(\vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)\right|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2}\frac{\mathrm{d}^2\vec{v}}{\mathrm{d}t^2}\cdot\vec{v}\,\mathrm{d}t\ . \tag{35}$$

So, energy conservation holds if

$$\int_{t_1}^{t_2}\left(\vec{F}_{\mathrm{rad}} - \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}\right)\cdot\vec{v}\,\mathrm{d}t = 0\ , \tag{36}$$

which will hold if

$$\boxed{\vec{F}_{\mathrm{rad}} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}\ .} \tag{37}$$

$$\boxed{\vec{F}_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}} \tag{37}$$

is the Abraham-Lorentz formula for the radiation reaction force. We have not proven it, but we have shown that it provides a force that is consistent with conservation of energy, at least for periodic motion.

The Abraham-Lorentz formula can be "derived" by modeling the point particle as an extended object, so that one can calculate the force that one part of the object exerts on the other parts. One then takes a limit in which the size of the extended object approaches zero. But the result depends on the details of the model, so the result remains inconclusive.

The Abraham-Lorentz formula gives reasonable predictions for the radiation reaction force under many circumstances, but it also has an important pathology: runaway solutions. Note that

$$\vec{F}_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}} = m\vec{a} \tag{38}$$

has the solution

$$a(t) = a_0\, e^{t/\tau} \ , \tag{39}$$

where

$$\tau = \frac{\mu_0 q^2}{6\pi m c} \approx 6 \times 10^{-24} \text{ s} \tag{40}$$

for an electron.

In quantum electrodynamics the situation is at least under much better control. Infinities arise, but they can be removed from the theory by a technique called renormalization. This process can be proven to succeed order by order in perturbation theory (an expansion in powers of the $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$), but the perturbation expansion is not believed to converge. Nothing can be proven rigorously, but there are reasons to believe that the theory is not completely well-defined, but that it nonetheless gives a consistent description up to extraordinarily high energies, well beyond the Planck mass.

8.07 Electromagnetism II
Fall 2012