Physics 8.07: Electromagnetism II   September 15, 2012
Prof. Alan Guth

PROBLEM SET 2

DUE DATE: Monday, September 24, 2012. Either hand it in at the lecture, or by 6:00 pm in the 8.07 homework boxes.

READING ASSIGNMENT: Chapter 2 of Griffiths: *Electrostatics.*

GRADING OF THIS PROBLEM SET: The problem set is worth 85 points plus 20 points extra credit. It is therefore time to clarify the operational definition of "extra credit". We will keep track of the extra credit grades separately, and at the end of the course I will first assign provisional grades based solely on the regular coursework. I will consult with Prof. Chen and Ahmet Demir, and we will try to make sure that these grades are reasonable. Then I will add in the extra credit, allowing the grades to change upwards accordingly. Finally, we will look at each student's grades individually, and we might decide to give a higher grade to some students who are slightly below a borderline. Students whose grades have improved significantly during the term, and students whose average has been pushed down by single low grade, will be the ones most likely to be boosted.

    The bottom line is that you should feel free to skip the extra credit problems, and you will still get an excellent grade in the course if you do well on the regular problems. However, if you are the kind of student who really wants to get the most out of the course, then I hope that you will find these extra credit problems challenging, interesting, and educational. (As described in the solutions to Problem Set 1, the problem sets in the course will not be graded in full, but instead only selected problems will actually be graded. The extra credit problems will never be among the graded problems.)

PROBLEM 1: THE LAPLACIAN AS THE ANTI-LUMPINESS OPERA-TOR *(15 points)*

    In this problem you will prove a relation that was stated in lecture. Le $\varphi(\vec{r})$ be any scalar function of position $\vec{r}$. We are interested in relating the value of φ at an arbitrary point $\vec{r}_0$ to the average value of φ on a sphere that is centered at $\vec{r}_0$. While the point $\vec{r}_0$ is arbitrary, we can simpify our notation by choosing a coordinate system so that $\vec{r}_0$ is the origin $\vec{0}$. Then the relation to be proved can be written

$$\varphi(\vec{0}) - \bar{\varphi}(R) = -\frac{1}{4\pi} \int_{r<R} \mathrm{d}^3x \left(\frac{1}{r} - \frac{1}{R} \right) \nabla^2 \varphi \ . \tag{1.1}$$

Here $\bar{\varphi}(R)$ represents the average value of φ on the surface of a sphere of radius R, which can be written explicitly as

$$\bar{\varphi}(R) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \varphi(R, \theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{1.2}$$

The integration in Eq. (1.1) is over the volume of a sphere of radius R, centered at the origin. The relation to lumpiness can be seen by thinking of φ as the density of a pudding. The equation implies that if $\nabla^2 \varphi = 0$, then the value of φ at the origin is the same as the average value of its surroundings (no lumpiness). But if $\nabla^2 \varphi < 0$, then the value of φ at the origin is higher than the average value of its surroundings (i.e., there is a lump).

(a) Defining

$$g(r) \equiv \frac{1}{r} - \frac{1}{R} \tag{1.3}$$

for compactness, use the divergence theorem to show that

$$\int_{r<R} d^3x \, \vec{\nabla} \cdot \left(g \, \vec{\nabla} \varphi \right) = 0 \ . \tag{1.4}$$

(b) Use index notation (i.e., $\vec{\nabla} \equiv \hat{e}_i \partial_i$) to show that for arbitrary scalar functions $g(\vec{r})$ and $\varphi(\vec{r})$

$$\vec{\nabla} \cdot \left(g \, \vec{\nabla} \varphi \right) = \vec{\nabla} g \cdot \vec{\nabla} \varphi + g \, \nabla^2 \varphi \ . \tag{1.5}$$

(c) Use the identity in part (b) to rewrite the integrand of the integral of Eq. (1.4), and evaluate each term separately. We suggest that the integral of $\vec{\nabla} g \cdot \vec{\nabla} \varphi$ be expressed in spherical polar coordinates. Show that the vanishing of the integral in Eq. (1.4), re-expressed in this way, implies Eq. (1.1).

PROBLEM 2: CAPACITANCE OF A CYLINDRICAL CAPACITOR *(10 points)*

A very long conducting cylinder (length ℓ and radius a) carrying a total charge $+q$ is surrounded by a thin conducting cylindrical shell (length ℓ and radius b) with total charge $-q$, as shown in cross section in the sketch.
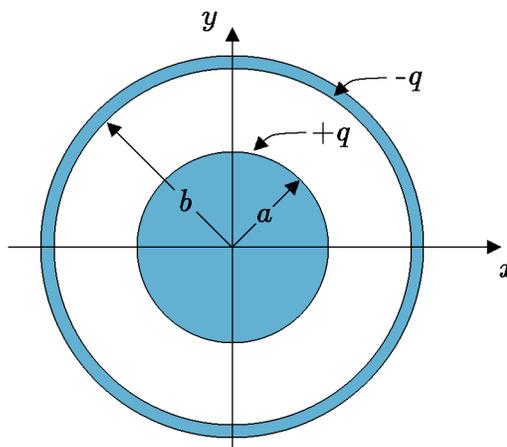
(a) Using Gauss's law, find an expression for the electric field $\vec{E}(\vec{r})$ at points $a < r < b$. Neglect end effects due to the finite length of the capacitor.

(b) Using your expression for $\vec{E}$ from part (a), find the potential difference ΔV between the outer shell and the inner cylinder.

(c) Derive an expression for the capacitance of this capacitor in terms of the quantities given. What is the capacitance per unit length?

(d) Let the gap $d = b - a$ between the cylinders be small compared to the radii, a and b. Show that in this case your answer for part (c) reduces to that for a parallel plate capacitor (see Griffiths Eq. (2.54) on p. 105).

PROBLEM 3: THE ELECTRIC FIELD, POTENTIAL, AND ENERGY OF A UNIFORM SPHERE OF CHARGE *(15 points)*

(a) A uniformly charged sphere of charge has radius R and total charge Q. Using Gauss's law, calculate the electric field $\vec{E}(\vec{r})$ everywhere.

(b) Using the electric field you calculated in part (a), find the electric potential $V(\vec{r})$ everywhere.

(c) Using the expression

$$W = \frac{1}{2}\epsilon_0 \int_{\text{all space}} |\vec{E}|^2 \, \mathrm{d}^3 x \;, \tag{3.1}$$

for the total work needed to assemble the charge configuration, calculate W using your expressions above.

(d) Using the expression

$$W = \frac{1}{2} \int_{\text{all space}} \rho V \, \mathrm{d}^3 x \;, \tag{3.2}$$

calculate W again using your expressions above.

PROBLEM 4: CALCULATING FORCES USING VIRTUAL WORK *(10 points)*

Use "virtual work" to calculate the attractive force between conductors in the parallel plate capacitor (area A, separation d). That is, use conservation of energy to determine how much work must be done to move one plate by an infinitesimal amount, and then use the value of the work to determine the force. Do your virtual work computations in two ways:

(a) keeping fixed the charges on the plates, and,

(b) keeping a fixed the voltage between the plates.

PROBLEM 5: MUTUAL CAPACITANCE *(15 points)*

In lecture we discussed relations of the form

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j, \quad i, j = 1, 2, \ldots, n \;. \tag{5.1}$$

governing the potentials and charges of n conductors (with the potential taken to be zero at spatial infinity).

(a) Prove that $C_{ij} = C_{ji}$. [*Hint:* Consider how much energy is needed to start with the system uncharged, then add charge Q_i to conductor i, and then add charge Q_j to

conductor j. Then consider starting again with the system uncharged, and performing these operations in the opposite order. That is, add charge Q_j to conductor j, and then Q_i to conductor i. Then think about how to use your answers to prove the desired result.]

(b) Consider a two-conductor configuration. Calculate the conventional capacitance C in terms of C_{11}, C_{12}, C_{21}, and C_{22}.

(c) Consider two concentric spherical conducting shells of radii a and b with $a < b$. Call the inner shell conductor 1, and the outer shell conductor 2. Calculate the matrix of capacitances C_{ij} and use your result from part (b) to infer the conventional capacitance C. Compare your answer with Example 2.11 in Griffiths, p. 105.

PROBLEM 6: SPACE CHARGE, VACUUM DIODES, AND THE CHILD-LANGMUIR LAW *(20 points)*

Griffiths Problem 2.48 (p. 107). Challenging! For part (e), you can solve the differential equation either by guessing a solution and showing that it works, or by finding a first integral by the same method that is used in mechanics to go from Newton's 2nd order equation of motion to the first order equation for the conservation of mechanical energy.

PROBLEM 7: $\nabla^2(1/r)$ IN THE LANGUAGE OF DISTRIBUTIONS *(20 points extra credit)*

This problem will have a longwinded pedagogical introduction, since it concerns an approach which was discussed in lecture, but is not discussed in the textbook.

In Problem 5 of Problem Set 1, you evaluated $-\nabla^2(1/4\pi r)$ by replacing $1/r$ by $1/\sqrt{r^2 + a^2}$. After calculating $g_a(r) \equiv -\nabla^2(1/4\pi\sqrt{r^2 + a^2})$, you showed that its integral over all space is 1, and that for any $r \neq 0$ it approaches 0 as $a \to 0$. This exercise was intended to convey a useful intuition about δ-functions, and about the relation

$$-\nabla^2 \frac{1}{4\pi r} = \delta^3(\vec{r}) \ . \tag{7.1}$$

However, from the standpoint of a mathematically rigorous treatment, there is a shortcoming to this and all similar treatments of the δ-function as a limit of a sequence of functions. While the sequence of functions leads to reliable intuition, the precise mathematical picture is complicated by the ordering of limits. That is, you showed in your problem set solutions that $\int_{\text{all space}} g_a(r)\, \mathrm{d}^3x = 1$ for any $a > 0$, and hence

$$\lim_{a \to 0} \int_{\text{all space}} g_a(r)\, \mathrm{d}^3x = 1 \ . \tag{7.2}$$

However, if we had taken the limit first, we would have found

$$\lim_{a \to 0} g_a(r) = \begin{cases} 0 & \text{if } r \neq 0 \\ \infty & \text{if } r = 0 \end{cases} , \tag{7.3}$$

and we showed in lecture that the integral of this function, defined as the area under the curve, is in fact zero. So we cannot quite say that $g_a(r)$ approaches a δ-function as $a \to 0$. Instead, we have to keep in mind the slightly more complicated picture in which $g_a(r)$ acts like a δ-function when a is very very small, and behaves exactly as a δ-function if we take the limit $a \to 0$ *after* any integrations have been carried out.

Since the integral of the function described in Eq. (7.3) vanishes, there is no normal function that behaves as a Dirac δ-function. Thus the δ-function is technically not a function, but rather what the mathematicians call a *generalized function*, or a *distribution*. It is really the concept of integration that is being generalized, and a distribution is the integrand of a generalized integral. Starting with functions of one variable, we can consider an arbitrary function $\varphi(x)$. Its integral,

$$\int_{-\infty}^{\infty} \varphi(x) \, dx , \tag{7.4}$$

maps the function $\varphi(x)$ into a single real number, the value of its integral. It is a linear map, in the sense that

$$\int_{-\infty}^{\infty} [\varphi_1(x) + \lambda \varphi_2(x)] \, dx = \int_{-\infty}^{\infty} \varphi_1(x) \, dx + \lambda \int_{-\infty}^{\infty} \varphi_2(x) \, dx , \tag{7.5}$$

where λ is a constant. A distribution defines a generalized integral, which is an *arbitrary* linear map from the space of smooth "test" functions $\varphi(x)$ to real numbers. These test functions are required not only to be smooth, but also to fall off rapidly at large values of $|x|$.* The distribution that corresponds to a δ-function is the map which takes the function $\varphi(x)$ to $\varphi(x_0)$, its value at some particular point x_0. While there is no function that behaves as a Dirac δ-function, it is perfectly clear that this map from functions to real numbers is well-defined. Thinking of this map as a generalization of integration, we can write it as

$$\int \varphi(x) \, \delta(x - x_0) \, dx \equiv \varphi(x_0) . \tag{7.6}$$

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 * Various choices can be made for the precise restrictions on the space of test functions. A frequently used choice is the space of Schwartz functions, which are infinitely differentiable, and which have the property that the function and all its derivatives fall off faster than any power at large $|x|$. The distributions associated with this definition of smoothness are called tempered distributions.

But remember that the integral sign here does not describe the area under a curve; instead it denotes a linear map from the function $\varphi(x)$ to a real number, and the symbol $\delta(x - x_0)$ indicates the particular linear map which maps $\varphi(x)$ to its value at x_0, namely $\varphi(x_0)$. Mathematically, Eq. (7.6) defines the δ-function, which is defined solely as a prescription for a generalized type of integration.

The derivative of a distribution is defined so that generalized integration is consistent with the usual procedure of integration by parts:

$$\int \varphi(x) \frac{\mathrm{d}}{\mathrm{d}x} \delta(x - x_0) \, \mathrm{d}x \equiv - \int \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} \delta(x - x_0) \, \mathrm{d}x \equiv -\varphi'(x_0) \ , \tag{7.7}$$

where $\varphi'(x) \equiv \mathrm{d}\varphi(x)/\mathrm{d}x$. Note that we do not include any boundary terms, as $\varphi(x)$ is required to fall off at large $|x|$ fast enough to cause any boundary terms to vanish.

We are now ready to evaluate $\nabla^2(1/r)$ in the language of distributions. Note that in the language of functions $\nabla^2(1/r)$ is ill-defined, because $1/r$ is not differentiable at $r = 0$. But we can promote $1/r$ to a distribution by defining it as a mapping from test functions $\varphi(\vec{r})$ to numbers, where the mapping is given by the (ordinary) integral

$$\int \varphi(\vec{r}) \frac{1}{r} \, \mathrm{d}^3 x \ . \tag{7.8}$$

Note that even though $1/r$ is singular at $r = 0$, this integral is perfectly well defined, since in spherical polar coordinates we have

$$\frac{1}{r} \, \mathrm{d}^3 x = r \, \mathrm{d}r \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \ . \tag{7.9}$$

By defining the derivative of a distribution by integration by parts, as in Eq. (7.7), we can write the distribution corresponding to $\nabla^2(1/r)$, which I will call $F[\varphi(\vec{r})]$ for future reference:

$$\int \varphi(\vec{r}) \nabla^2 \left(\frac{1}{r}\right) \mathrm{d}^3 x = \int \varphi(\vec{r}) \, \partial_i \partial_i \left(\frac{1}{r}\right) \mathrm{d}^3 x \tag{7.10a}$$

$$= - \int \partial_i \varphi(\vec{r}) \, \partial_i \left(\frac{1}{r}\right) \mathrm{d}^3 x \tag{710b}$$

$$= \int \nabla^2 \varphi(\vec{r}) \left(\frac{1}{r}\right) \mathrm{d}^3 x \ , \tag{7.10c}$$

so

$$F[\varphi(\vec{r})] \equiv \int \nabla^2 \varphi(\vec{r}) \left(\frac{1}{r}\right) \mathrm{d}^3 x \ . \tag{7.11}$$

Note that the square brackets used for the argument of F is a common notation for a functional, i.e., a function of a function. Here F maps the function $\varphi(\vec{r})$ to the number given by the (ordinary) integral on the right-hand side of Eq. (7.11).

AT LAST: THE HOMEWORK PROBLEM:

(a) Evaluate $F[\varphi(\vec{r})]$ (as defined by Eq. (7.11)) for an arbitrary smooth test function $\varphi(\vec{r})$ which falls off rapidly for large $|\vec{r}|$. Show that

$$F[\varphi(\vec{r})] = -4\pi\varphi(\vec{0}) \ . \tag{7.12}$$

Since

$$\int \varphi(\vec{r}) \, \delta^3(\vec{r}) = \varphi(\vec{0}) \ , \tag{7.13}$$

Eq. (7.12) is equivalent to writing

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r}) \tag{7.14}$$

in the sense of distributions, which is the result we seek. [*Hint:* Although Eq. (7.11) is the defining equation, there is nothing that prevents you from integrating by parts once to retrieve the integral in the form of Eq. (7.10b). Write this in spherical polar coordinates, and then try to evaluate it.]

(b) Use the language of distributions to evaluate $\nabla^2 \ln r$ in two dimensions. (See Problem 6(d) of Problem Set 1.)

8.07 Electromagnetism II
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