Quantum Physics III (8.06) — Spring 2016

Assignment 7

Readings

- Density matrices and decoherence are not well covered in any 8.06 textbook, so the lecture notes are more thorough on this topic. However, some additional optional readings are:
  - Sakurai, Section 3.4
  - Cohen-Tannoudji, Complements E_{III} and F_{IV}.
- Review 8.05 notes on tensor products and entanglement.

1. **Pure states (10 points)**
   Let ρ be a finite-dimensional density matrix. Recall that ρ is said to be a pure state if $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$. Prove that $\text{tr}(\rho^2) = 1$ if and only if ρ is pure.

2. **"Mercedes" states (10 points)**
   Write down three spin-1/2 states $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ such that if each occurs with probability 1/3, the resulting density operator is $\frac{I}{2}$.

3. **Gaussian phase error (10 points)**
   Consider an electron spin in the state

$$\rho = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix}$$

   that experiences a magnetic field $B\hat{\mathbf{z}}$. The Hamiltonian is then $H = -\gamma B \hat{S}_z$ with $\gamma = g_e e/2m_e$. Suppose that the field strength B is drawn from a Gaussian distribution with mean 0 and variance σ^2; i.e. the probability density of B is

$$f(B) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{B^2}{2\sigma^2}}.$$

   Let ρ' be the state that results from applying this field for time t and averaging over the possible values of B. Compute ρ'.

4. **Lasers vs light bulbs (20 points)**

   (a) The state of a laser is often described by a coherent state

   $$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

   where $|n\rangle$ is the number state with n photons. However, in practice, we may know $|\alpha|$ but will generally be ignorant of the phase of α. We can model this by thinking of α as a random variable of the form $re^{i\phi}$ where $r \geq 0$ is given and ϕ is uniformly random on the interval $[0, 2\pi]$. (In reality, even r might be incompletely known, but assume for the sake of this problem that we know r exactly.) Write down the resulting density operator ρ_{laser} in the number basis. What is $\langle \hat{n} \rangle_{\text{laser}}$ as a function of r?

   (b) By contrast, an incandescent light bulb produces light that is in a thermal state. Consider only light of a fixed angular frequency ω (i.e. of frequency $\nu = \omega/2\pi$). Write down the density operator for the thermal state ρ_{thermal} at temperature T in the number basis. Express this as a function of the dimensionless quantity $\gamma \equiv \hbar\omega/k_B T$. What is $\langle \hat{n} \rangle_{\text{thermal}}$? Here the "thermal state" refers to the density matrix corresponding to the canonical distribution, in which a state x with energy $E(x)$ has probability $e^{-\beta E(x)}/Z$ where $\beta = 1/k_B T$ and $Z = \sum_{x'} e^{-\beta E(x')}$.

   (c) By observing the average photon number $\langle \hat{n} \rangle$ alone it is impossible to distinguish the state of a laser from that of a thermal state. Suppose we instead measure fluctuations in photon number, i.e. $\Delta\hat{n}^2 \equiv (\hat{n} - \langle \hat{n} \rangle)^2$. Compute $\langle \Delta\hat{n}^2 \rangle_{\text{laser}}$ and $\langle \Delta\hat{n}^2 \rangle_{\text{thermal}}$. Using parts (a) and (b), express your answers in terms of $\langle \hat{n} \rangle$. Explain how this can be used to distinguish these two sources of light. [Hint: Using $\langle \Delta\hat{n}^2 \rangle = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$ may simplify your calculation.]

5. **Bloch equation (20 points)** This problem describes a spin-1/2 particle in a magnetic field undergoing thermal relaxation and dephasing noise. Given positive constants $\gamma, B, \beta, T_1, T_2$, let $H = -\gamma B S_z$ and $\rho_{\text{th}} = e^{-\beta H}/\text{tr}[e^{-\beta H}]$. Assume that the state of the system evolves according to

   $$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] - \frac{1}{T_1}(\rho - \rho_{\text{th}}) - \frac{1}{T_2}\begin{pmatrix} 0 & \rho_{+-} \\ \rho_{-+} & 0 \end{pmatrix}. \qquad (1)$$

   If $\rho = \frac{I + \vec{a}\cdot\vec{\sigma}}{2}$ (with $|\vec{a}| \leq 1$) then show that (1) can be expressed as

   $$\frac{\partial \vec{a}}{\partial t} = M\vec{a} + \vec{b}, \qquad (2)$$

   with M a 3×3 matrix and $\vec{b} \in \mathbb{R}^3$. Find $M, \vec{b}$. Solve this differential equation. Assuming that $T_1 \gg T_2 \gg 1/\gamma B$, briefly qualitatively explain the salient features of your solution, such as: Does it reach a steady state? What path does it take to get there? etc.

6. **Spontaneous emission (30 points)** Model an atom as a two-level system with ground state $|g\rangle$ and excited state $|e\rangle$. Suppose the atom interacts with a photon field (i.e. a harmonic oscillator) via the Hamiltonian

$$H = \hbar\Omega(|g\rangle\langle e| \otimes \hat{a}^\dagger + |e\rangle\langle g| \otimes \hat{a}). \tag{3}$$

(For a justification see problem 3 of pset 5. But for the purposes of this problem we will take (3) to be an assumption.) This problem will involve the following decoherence process:

(i) Add a photon field in state $|0\rangle\langle 0|$; i.e. map the state ρ to $\rho \otimes |0\rangle\langle 0|$.

(ii) Apply the Hamiltonian in (3) for time τ.

(iii) Discard the photon state.

(a) Suppose we apply the above decoherence process once. If the atom starts with density operator ρ, then explain why this leaves the atom with density operator

$$\rho' = \mathrm{tr}_{\mathrm{photon}}\left[e^{-\frac{iH\tau}{\hbar}}(\rho \otimes |0\rangle\langle 0|)e^{\frac{iH\tau}{\hbar}}\right].$$

Compute ρ' to order $O(\tau^2)$ (i.e. neglecting τ^3 and higher terms).

(b) Now imagine that we repeat the above three steps every τ seconds. We would like to approximate this process with a continuous-time evolution by taking $\tau \to 0$. In order to obtain a nontrivial answer, we will make Ω change with τ. Specifically suppose we take $\tau \to 0$ while holding $\delta \equiv \Omega^2\tau$ fixed. Derive a differential equation for ρ of the form

$$\dot{\rho} = L[\rho]$$

where $L[\rho]$ is a matrix-valued function of ρ that is not always zero. Equivalently,

$$\rho(t+\tau) = \rho(t) + L[\rho(t)]\tau + O(\tau^2),$$

where $L[\cdot]$ may be a function of δ but not (directly) τ. What is the steady-state solution of this differential equation? Is it unique?

[Note: The assumption that $\Omega^2 \sim 1/\tau$ is a crude approximation to what actually happens. In part (a), you found that the decoherence from coupling to a single photon mode was proportional to τ^2. However, the number of modes that couple to the atom at time τ scales as $1/\tau$. Summing over these yields a change in the state proportional to τ. Taking $\Omega^2 \sim 1/\tau$ is a simpler, but less justified, way of getting to the same conclusion.]

(c) Now modify the original process so that instead of adding a photon field in state $|0\rangle\langle 0|$ in the step (i), we add a thermal state with inverse temperature β. Assume here that the photons have angular frequency ω. Repeat the analysis in parts (a) and (b) of this problem to find the resulting differential equation for ρ. What is the equilibrium state for an atom undergoing this process?

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