Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2 \; ; \quad J(x,t) = \frac{\hbar}{2im}\left[\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \frac{\int dx\,\psi^*(x)H\psi(x)}{\int dx\psi^*(x)\psi(x)} \equiv \langle H\rangle_\psi \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach}: \quad H = -\boldsymbol{\mu}\cdot\mathbf{B}\,, \quad \boldsymbol{\mu} = g\,\frac{e\hbar}{2m}\,\frac{1}{\hbar}\mathbf{S} = \gamma\mathbf{S}$$

$$\mu_B = \frac{e\hbar}{2m_e}\,, \quad \boldsymbol{\mu}_e = -2\,\mu_B\frac{\mathbf{S}}{\hbar}\,,$$

$$\text{In the basis } |1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}\,, \quad |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\,; \quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\,; \quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad [S_i,S_j] = i\hbar\,\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\,I + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\,\mathbf{M}\sin\theta\,, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp(i\,\mathbf{a}\cdot\boldsymbol{\sigma}) = \mathbf{1}\cos a + i\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{a}}{a}\right)\sin a\,, \quad a = |\mathbf{a}|$$

$$\exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) = \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta)$$
$$\exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) = \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)\,.$$

$$S_{\mathbf{n}} = \mathbf{n}\cdot\mathbf{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\,.$$

$$(n_x,n_y,n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad S_{\mathbf{n}}|\mathbf{n};\pm\rangle = \pm\frac{\hbar}{2}|\mathbf{n};\pm\rangle$$

$$|\mathbf{n};+\rangle = \cos(\tfrac{1}{2}\theta)|+\rangle + \sin(\tfrac{1}{2}\theta)\exp(i\phi)|-\rangle$$

$$|\mathbf{n};-\rangle = -\sin(\tfrac{1}{2}\theta)\exp(-i\phi)|+\rangle + \cos(\tfrac{1}{2}\theta)|-\rangle$$

$$\left|\langle\mathbf{n}';+|\mathbf{n};+\rangle\right| = \cos(\tfrac{1}{2}\gamma)\,, \quad \gamma \text{ is the angle between } \mathbf{n} \text{ and } \mathbf{n}'$$

$$\langle\mathbf{S}\rangle_{\mathbf{n}} = \frac{\hbar}{2}\mathbf{n}\,, \quad \text{Rotation operator: } R_\alpha(\mathbf{n}) \equiv \exp\left(-\frac{i\alpha S_{\mathbf{n}}}{\hbar}\right)$$

- Linear algebra

$$\text{Matrix representation of } T \text{ in the basis } (v_1, \ldots, v_n): \quad Tv_j = \sum_i T_{ij}\, v_i$$

$$\text{basis change: } u_k = \sum_j A_{jk} v_j\,, \qquad T(\{u\}) = A^{-1}T(\{v\})A$$

$$\text{Schwarz: } |\langle u, v\rangle| \leq |u|\,|v|$$

$$\text{Adjoint: } \langle u, Tv\rangle = \langle T^\dagger u, v\rangle\,, \quad (T^\dagger)^\dagger = T$$

- Bras and kets: For an operator Ω and a vector v, we write $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint: } \langle u|\Omega^\dagger v\rangle = \langle \Omega u|v\rangle$$

$$|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle \quad \longleftrightarrow \quad \langle \alpha_1 v_1 + \alpha_2 v_2| = \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}\,, \qquad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\Omega_{ij} = \langle i|\Omega|j\rangle \quad \leftrightarrow \quad \Omega = \sum_{i,j}\Omega_{ij}\,|i\rangle\langle j|$$

$$\langle i|\Omega^\dagger|j\rangle = \langle j|\Omega|i\rangle^*$$

$$\Omega \text{ hermitian: } \Omega^\dagger = \Omega, \qquad U \text{ unitary: } U^\dagger = U^{-1}$$

- Matrix M is normal $([M, M^\dagger] = 0)$ $\longleftrightarrow$ unitarily diagonalizable.

- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \langle x|y\rangle = \delta(x - y)\,, \quad \mathbf{1} = \int dx\,|x\rangle\langle x|\,, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle\,, \quad \langle q|p\rangle = \delta(q - p)\,, \quad \mathbf{1} = \int dp\,|p\rangle\langle p|\,, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right)\,; \qquad \tilde{\psi}(p) = \int dx\langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\exp\left(-\frac{ipx}{\hbar}\right)\psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \left(\frac{\hbar}{i}\frac{d}{dx}\right)^n\psi(x)\,; \qquad \langle p|\hat{x}^n|\psi\rangle = \left(i\hbar\frac{d}{dp}\right)^n\tilde{\psi}(p)\,; \qquad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i}f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)dx = \delta(k)$$

- Generalized uncertainty principle

$$\Delta A \equiv |(A - \langle A \rangle \mathbf{1})\Psi| \quad \rightarrow \quad (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$$

$$\Delta A \, \Delta B \geq \left| \langle \Psi | \frac{1}{2i}[A, B] | \Psi \rangle \right|$$

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \quad \text{and} \quad \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \quad \text{for} \quad \psi \sim \exp\left(-\frac{1}{2}\frac{x^2}{\Delta^2}\right)$$

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\text{Time independent operator } Q: \quad \frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [H, Q] \rangle$$

$$\Delta H \Delta t \geq \frac{\hbar}{2}, \quad \Delta t \equiv \frac{\Delta Q}{\left| \frac{d\langle Q \rangle}{dt} \right|}$$

- Commutator identities

$$[A, BC] = [A, B]C + B[A, C],$$

$$e^A B e^{-A} = e^{\mathrm{ad}_A} B = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots,$$

$$e^A B e^{-A} = B + [A, B], \quad \text{if} \quad [A, [A, B]] = 0,$$

$$[B, e^A] = [B, A]e^A, \quad \text{if} \quad [A, [A, B]] = 0$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{\frac{1}{2}[A,B]}, \quad \text{if } [A, B] \text{ commutes with } A \text{ and with } B$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right), \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right),$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a}),$$

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger.$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad \hat{N}|n\rangle = n|n\rangle, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle\,,\quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\,.$$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\,.$$

$$x_H(t) = \hat{x}\,\cos\omega t + \frac{\hat{p}}{m\omega}\,\sin\omega t$$

$$p_H(t) = \hat{p}\,\cos\omega t - m\omega\,\hat{x}\,\sin\omega t$$

- Coherent states and squeezed states

$$T_{x_0} \equiv e^{-\frac{i}{\hbar}\hat{p}\,x_0}\,,\quad T_{x_0}|x\rangle = |x+x_0\rangle$$

$$|\tilde{x}_0\rangle \equiv T_{x_0}|0\rangle = e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle\,,$$

$$|\tilde{x}_0\rangle = e^{-\frac{1}{4}\frac{x_0^2}{d^2}}e^{\frac{x_0}{\sqrt{2}d}a^{\dagger}}|0\rangle\,,\quad \langle x|\tilde{x}_0\rangle = \psi_0(x-x_0)\,,\quad d^2 = \frac{\hbar}{m\omega}$$

$$|\alpha\rangle \equiv D(\alpha)|0\rangle = e^{\alpha a^{\dagger}-\alpha^* a}|0\rangle\,,\quad D(\alpha) \equiv \exp\left(\alpha a^{\dagger}-\alpha^* a\right)\,,\quad \alpha = \frac{\langle\hat{x}\rangle}{\sqrt{2}\,d} + i\,\frac{\langle\hat{p}\rangle\,d}{\sqrt{2}\,\hbar} \in \mathbb{C}$$

$$|\alpha\rangle = e^{\alpha a^{\dagger}-\alpha^* a}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\,a^{\dagger}}|0\rangle\,,\quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,,\quad |\alpha,t\rangle = e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle$$

$$|0_\gamma\rangle = S(\gamma)|0\rangle\,,\quad S(\gamma) = \exp\left(-\frac{\gamma}{2}(a^{\dagger}a^{\dagger}-aa)\right)\,,\quad \gamma \in \mathbb{R}$$

$$|0_\gamma\rangle = \frac{1}{\sqrt{\cosh\gamma}}\exp\left(-\frac{1}{2}\tanh\gamma\,a^{\dagger}a^{\dagger}\right)|0\rangle$$

$$S^{\dagger}(\gamma)\,a\,S(\gamma) = \cosh\gamma\,a - \sinh\gamma\,a^{\dagger}\,,\quad D^{\dagger}(\alpha)\,a\,D(\alpha) = a + \alpha$$

$$|\alpha,\gamma\rangle \equiv D(\alpha)S(\gamma)|0\rangle$$

- Time evolution

$$|\Psi,t\rangle = U(t,0)|\Psi,0\rangle\,,\quad U\ \text{unitary}$$

$$U(t,t) = \mathbf{1}\,,\quad U(t_2,t_1)U(t_1,t_0) = U(t_2,t_0)\,,\quad U(t_1,t_2) = U^{\dagger}(t_2,t_1)$$

$$i\hbar\frac{d}{dt}|\Psi,t\rangle = \hat{H}(t)|\Psi,t\rangle \quad\leftrightarrow\quad i\hbar\frac{d}{dt}U(t,t_0) = \hat{H}(t)U(t,t_0)$$

Time independent $\hat{H}$: $\quad U(t,t_0) = \exp\left[-\frac{i}{\hbar}\hat{H}(t-t_0)\right] = \sum_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|n\rangle\langle n|$

$$\langle A\rangle = \langle\Psi,t|A_S|\Psi,t\rangle = \langle\Psi,0|A_H(t)|\Psi,0\rangle \quad\rightarrow\quad A_H(t) = U^{\dagger}(t,0)A_S U(t,0)$$

$$[A_S, B_S] = C_S \quad\rightarrow\quad [A_H(t), B_H(t)] = C_H(t)$$

$$i\hbar\frac{d}{dt}\hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)]\,,\quad \text{for } A_S \text{ time-independent}$$

- Two state systems

$$H = h_0 \mathbf{1} + \mathbf{h} \cdot \boldsymbol{\sigma} = h_0 \mathbf{1} + h\,\mathbf{n} \cdot \boldsymbol{\sigma}\,, \quad h = |\mathbf{h}|$$

$$\text{Eigenstates: } |\mathbf{n}; \pm\rangle\,, \quad E_\pm = h_0 \pm h\,.$$

$$H = -\gamma\,\mathbf{S} \cdot \mathbf{B} \quad \rightarrow \quad \text{spin vector } \vec{n} \text{ precesses with Larmor frequency } \boldsymbol{\omega} = -\gamma \mathbf{B}$$

$$\text{NMR magnetic field } \mathbf{B} = B_0 \mathbf{z} + B_1 \big(\cos \omega t\,\mathbf{x} - \sin \omega t\,\mathbf{y}\big)$$

$$|\Psi(t)\rangle = \exp\Big(\frac{i}{\hbar}\omega t \hat{S}_z\Big) \exp\Big(\frac{i}{\hbar}\gamma \mathbf{B}_{\mathrm{R}} \cdot \mathbf{S}\,t\Big)|\Psi(0)\rangle$$

$$\mathbf{B}_{\mathrm{R}} = B_0\big(1 - \frac{\omega}{\omega_0}\big)\mathbf{z} + B_1\,\mathbf{x}$$

- Orbital angular momentum operators

$$\mathbf{a} \cdot \mathbf{b} \equiv a_i\, b_i\,, \quad (\mathbf{a} \times \mathbf{b})_i \equiv \epsilon_{ijk}\, a_j\, b_k\,, \quad \mathbf{a}^2 \equiv \mathbf{a} \cdot \mathbf{a}\,.$$

$$\epsilon_{ijk}\,\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\,, \qquad \epsilon_{ijk}\,\epsilon_{ijq} = 2\delta_{kq}\,.$$

$$\hat{L}_i = \epsilon_{ijk}\,\hat{x}_j\,\hat{p}_k \quad \Longleftrightarrow \quad \mathbf{L} = \mathbf{r} \times \mathbf{p} = -\mathbf{p} \times \mathbf{r}$$

Vector $\mathbf{u}$ under rotation: $\quad [\hat{L}_i\,,\,\hat{u}_j] = i\hbar\,\epsilon_{ijk}\,\hat{u}_k \quad \Longrightarrow \quad \mathbf{L} \times \mathbf{u} + \mathbf{u} \times \mathbf{L} = 2i\hbar\,\mathbf{u}$

Scalar S under rotation: $\quad [\hat{L}_i\,,\,S] = 0$

$\mathbf{u},\ \mathbf{v}$ vectors under rotations $\quad \rightarrow \quad \mathbf{u} \cdot \mathbf{v}$ is a scalar, $\quad \mathbf{u} \times \mathbf{v}$ is a vector

$$[\hat{L}_i\,,\,\hat{L}_j] = i\hbar\,\epsilon_{ijk}\,\hat{L}_k \quad \Longleftrightarrow \quad \mathbf{L} \times \mathbf{L} = i\hbar\,\mathbf{L}\,, \quad [\hat{L}_i\,,\,\mathbf{L}^2] = 0\,.$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}\,; \quad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta, \phi) \equiv \langle \theta, \phi | \ell, m\rangle$$

$$Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}\,; \quad Y_{1,\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\,; \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

- Algebra of angular momentum operators **J** (orbital or spin, or sum)

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k \quad \Longleftrightarrow \quad \mathbf{J} \times \mathbf{J} = i\hbar\, \mathbf{J}\,; \qquad [\mathbf{J}^2, J_i] = 0$$

$$\mathbf{J}^2 |jm\rangle = \hbar^2\, j(j+1)|jm\rangle\,; \qquad J_z |jm\rangle = \hbar m |jm\rangle\,, \quad m = -j, \ldots, j\,.$$

$$J_\pm = J_x \pm i J_y\,, \quad (J_\pm)^\dagger = J_\mp \quad J_x = \frac{1}{2}(J_+ + J_-)\,, \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$[J_z\,, J_\pm] = \pm\hbar\, J_\pm\,, \quad [J^2, J_\pm] = 0\,; \qquad [J_+, J_-] = 2\hbar\, J_z$$

$$\mathbf{J}^2 = J_+ J_- + J_z^2 - \hbar J_z = J_- J_+ + J_z^2 + \hbar J_z$$

$$J_\pm |jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

- Angular momentum in the two-dimensional oscillator

$$\hat{a}_L = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y)\,, \ \ \hat{a}_R = \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)\,, \ \ [\hat{a}_L, \hat{a}_L^\dagger] = [\hat{a}_R\,, \hat{a}_R^\dagger] = 1$$

$$J_+ = \hbar\, \hat{a}_R^\dagger \hat{a}_L\,, \quad J_- = \hbar\, \hat{a}_L^\dagger \hat{a}_R\,, \quad J_z = \tfrac{1}{2}\hbar(\hat{N}_R - \hat{N}_L)$$
$$|j, m\rangle: \qquad j = \tfrac{1}{2}(N_R + N_L)\,, \quad m = \tfrac{1}{2}(N_R - N_L)$$

$$\mathcal{H} = 0 \oplus \tfrac{1}{2} \oplus 1 \oplus \tfrac{3}{2} \oplus \ldots$$

- Radial equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2\, \ell(\ell+1)}{2mr^2}\right) u_{\nu\ell}(r) = E_{\nu\ell} u_{\nu\ell}(r) \quad \text{(bound states)}$$

$$u_{\nu\ell}(r) \sim r^{\ell+1}\,, \quad \text{as } r \to 0\,.$$

- Hydrogen atom

$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2}\,, \qquad \psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r} Y_{\ell,m}(\theta, \phi)$$

$$n = 1, 2, \ldots\,, \quad \ell = 0, 1, \ldots, n-1\,, \quad m = -\ell, \ldots, \ell$$

$$a_0 = \frac{\hbar^2}{me^2}\,, \qquad \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \qquad \hbar c \simeq 200\,\text{MeV–fm}$$

$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}}\exp(-r/a_0)$$

$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1 - \frac{r}{2a_0}\right)\exp(-r/2a_0)$$

$$u_{2,1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp(-r/2a_0)$$

- Addition of Angular Momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J_2}$

Uncoupled basis : $\quad |j_1 j_2; m_1 m_2\rangle \;$ CSCO : $\{\mathbf{J}_1^2, \mathbf{J}_2^2, J_{1z}, J_{2z}\}$

Coupled basis : $\qquad |j_1 j_2; jm\rangle \qquad$ CSCO : $\{\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}^2, J_z\}$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \ldots \oplus |j_1 - j_2|$$

$$|j_1 j_2; jm\rangle = \sum_{m_1 + m_2 = m} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle}_{\text{Clebsch-Gordan coefficient}}$$

$$\mathbf{J}_1 \cdot \mathbf{J}_2 = \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+}) + J_{1z}J_{2z}$$

Combining two spin $1/2$: $\quad \dfrac{1}{2} \otimes \dfrac{1}{2} = 1 \oplus 0$

$|1,1\rangle = |\uparrow\uparrow\rangle,$

$|1,0\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \qquad |0,0\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$|1,-1\rangle = |\downarrow\downarrow\rangle.$

8.05 Quantum Physics II
Fall 2013