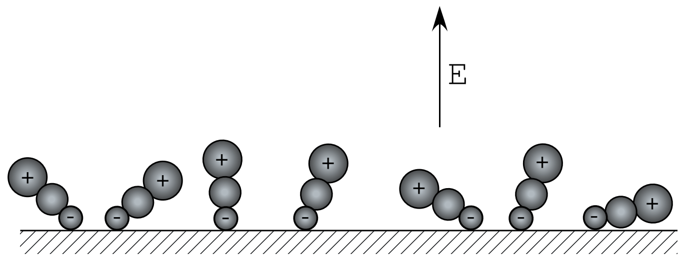
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2004

Exam #2

Problem 1 (25 points) Polar Molecules



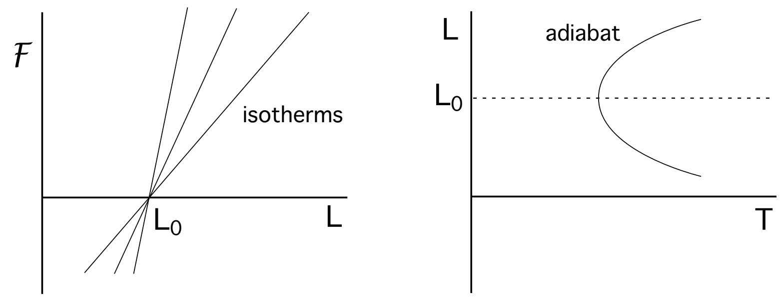
In a particular situation polar molecules (molecules possessing a permanent electric dipole moment) can be adsorbed on a surface creating a dipole layer with a total electric dipole moment $\mathcal{P}$ that remains finite even when the electric field perpendicular to the surface $\mathcal{E}$ goes to zero. Expressions for two important response functions in this system are given below.

$$\chi_T \equiv \left.\frac{\partial \mathcal{P}}{\partial \mathcal{E}}\right|_T = (a + \frac{b}{T})N + 3cN\mathcal{E}^2$$

$$\left.\frac{\partial T}{\partial \mathcal{E}}\right|_{\mathcal{P}} = \frac{aT^2 + bT + 3cT^2\mathcal{E}^2}{b\mathcal{E} - dT^2}$$

In these expressions a, b, c and d are constants and N is the number of molecules. One also knows that $\mathcal{P} = \mathcal{P}_0$ when $T = T_0$ and $\mathcal{E} = 0$. Find an analytic expression for the electric dipole moment $\mathcal{P}$.

Problem 2 (40 points) Elastic Rod



The internal energy U and the tension $\mathcal{F}$ in a certain elastic rod are given by the expressions

$$
\begin{aligned}
U(T,L) &= \frac{cT^4}{4} + \frac{a}{2}(L-L_0)^2 \\
\mathcal{F}(T,L) &= (a+bT)\,(L-L_0)
\end{aligned}
$$

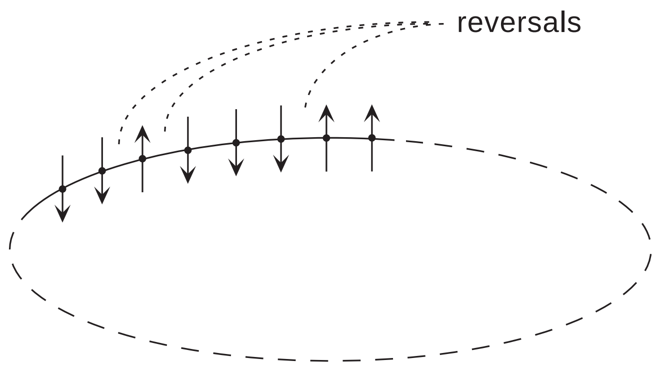
where a, b, c and L_0 are constants.

a) Find the work done on the rod, ΔW, as its length is doubled from L_0 to $2L_0$ along an isotherm at temperature T.

b) Find the heat added to the rod, ΔQ, along the same path as in a).

c) Find the differential equation $\dfrac{dL}{dT} = f(L,T)$ governing an adiabatic path in the $L-T$ plane. [Hint: you may want to check to see if your result is consistent with the sketch given above.]

Problem 3 (35 points) One-dimensional Ising Model



N spins are equally spaced around a circle in the x-y plane. Each spin can point either parallel or antiparallel to the z direction. There is no applied magnetic field so neither orientation is preferred. However, the spins interact with each other through a nearest neighbor interaction. If two neighboring spins point in the same direction, they contribute an amount $-J$ to the total energy; if they point in opposite directions, they contribute an amount J. Thus the total energy of the system depends on the number of reversals, R, that occur around the ring.

$$E = JR - J(N - R) = J(2R - N)$$

a) Assume that N is even. What are the smallest and largest values that R can have? What are the minimum and maximum values of E?

b) Find the total number of microscopic states of the system consistent with a given number of reversals, $\Omega(R)$. Note that this corresponds to the number of ways the R reversals can be distributed among the N inter-spin locations.

c) Assume that N is large. Find the entropy S of the spin chain as a function of N and R.

d) Find the energy of the spin chain as a function of temperature, $E(T)$. Make a sketch of the resulting function for the case $J > 0$ and indicate the low and high temperature asymptotes. Consider only positive T.

e) What is the mean value of R in the high temperature limit?

PARTIAL DERIVATIVE RELATIONSHIPS

Let x, y, z be quantities satisfying a functional relation $f(x,y,z) = 0$. Let w be a function of any two of x, y, z. Then

$$\left(\frac{\partial x}{\partial y}\right)_w \left(\frac{\partial y}{\partial z}\right)_w = \left(\frac{\partial x}{\partial z}\right)_w$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

COMBINATORIAL FACTS

There are $K!$ different orderings of K objects. The number of ways of choosing L objects from a set of K objects is

$$\frac{K!}{(K-L)!}$$

if the order in which they are chosen matters, and

$$\frac{K!}{L!(K-L)!}$$

if order does not matter.

STERLING'S APPROXIMATION

$$\ln K! \approx K \ln K - K \quad \text{when} \quad K >> 1$$

DERIVATIVE OF A LOG

$$\frac{d}{dx} \ln u(x) = \frac{1}{u(x)} \frac{du(x)}{dx}$$

LIMITS

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$$

$$\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

WORK IN SIMPLE SYSTEMS

System	Intensive quantity	Extensive quantity	Work
Hydrostatic system	P	V	$-PdV$
Wire	$\mathcal{F}$	L	$\mathcal{F}dL$
Surface	$\mathcal{S}$	A	$\mathcal{S}dA$
Reversible cell	E	Z	EdZ
Dielectric material	$\mathcal{E}$	$\mathcal{P}$	$\mathcal{E}d\mathcal{P}$
Magnetic material	H	M	HdM