MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I Spring Term 2013

Solutions, Exam #2

Problem 1 (30 points) Entropy of a Surface Film

γ and C_A are given in terms of T and A so it is reasonable to choose T and A as the variables in which to expand the entropy.

$$dS = \left(\frac{\partial S}{\partial T}\right)_A dT + \left(\frac{\partial S}{\partial A}\right)_T dA$$

$$C_A \equiv \left.\frac{đQ}{dt}\right|_A = T\left(\frac{\partial S}{\partial T}\right)_A \Rightarrow \left(\frac{\partial S}{\partial T}\right)_A = \frac{C_A}{T} = \frac{Nk_B}{T} + \frac{Nk_B T}{T_0^2}$$

To find $(\partial S/\partial A)_T$ use a Maxwell Relation. You may either use the magic square or derive the required relation as follows.

$$F \equiv U - TS$$

$$dF = -SdT + \gamma dA$$

cross derivatives of the prefactors of the differentials are equal

$$\left(\frac{\partial S}{\partial A}\right)_T = -\left(\frac{\partial \gamma}{\partial T}\right)_A = \frac{Nk_B}{A - bN}$$

Substituting in these results gives

$$
\begin{aligned}
dS &= \left(\frac{Nk_B}{T} + \frac{Nk_B T}{T_0^2}\right) dT + \left(\frac{Nk_B}{A - bN}\right) dA \\
S &= Nk_B \ln T + \frac{1}{2} Nk_B \left(\frac{T}{T_0}\right)^2 + f(A) \\
\left(\frac{\partial S}{\partial A}\right)_T &= f'(A) = \frac{Nk_B}{A - bN} \Rightarrow f(A) = Nk_B \ln(A - bN) + c \\
S(T, A) &= \underline{Nk_B \ln T + \frac{1}{2} Nk_B \left(\frac{T}{T_0}\right)^2 + Nk_B \ln(A - bN) + c}
\end{aligned}
$$

[Note: One can make the arguments of the logs dimensionless by distributing part of the additive constant c among the various other terms.]

$$
S(T, A) = \underline{Nk_B \ln(T/T_1) + \frac{1}{2} Nk_B \left(\frac{T}{T_0}\right)^2 + Nk_B \ln((A - bN)/A_1) + c'}
$$

Problem 2 (40 points) Crystal Field Splitting

a)

$$Z_1 = 1 + 2\exp[-\Delta/k_BT]$$

$$<\epsilon> = \sum_{\text{states}} \epsilon_{\text{state}}\, p(\text{state}) = \frac{2\Delta\exp[-\Delta/k_BT]}{1+2\exp[-\Delta/k_BT]} = 2\Delta\,\frac{1}{\exp[\Delta/k_BT]+2}$$

$$U(T,N) = N<\epsilon> = \underline{2\Delta N\,\frac{1}{\exp[\Delta/k_BT]+2}}$$

b) At $T = 0$ only the non-degenerate ground state is occupied. $\underline{S(T=0,N) = k_BN\ln(1) = 0.}$

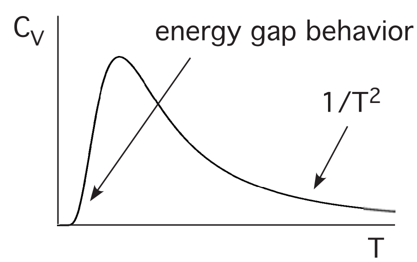
As $T \to \infty$, all three states are equally probable. $\underline{S(T,N) \to k_BN\ln(3).}$

c)

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 2\Delta N\frac{d}{dT}\left(\frac{1}{\exp[\Delta/k_BT]+2}\right)$$

$$= 2\Delta N\frac{(\Delta/k_BT^2)\exp[\Delta/k_BT]}{(\exp[\Delta/k_BT]+2)^2}$$

$$= \underline{2Nk_B\left(\frac{\Delta}{k_BT}\right)^2\frac{\exp[\Delta/k_BT]}{(\exp[\Delta/k_BT]+2)^2}}$$



d)

$$F(T,N) = -k_BT\ln Z = -Nk_BT\ln Z_1 = -Nk_BT\ln(1+2\exp[-\Delta/k_BT])$$

$$P(T,N) = -\left(\frac{\partial F}{\partial V}\right)_T = -\left(\frac{\partial F}{\partial \Delta}\right)_T\frac{d\Delta}{dV} = \gamma\left(\frac{\Delta}{V}\right)\left(\frac{\partial F}{\partial \Delta}\right)_T$$

$$= -Nk_BT\gamma\left(\frac{\Delta}{V}\right)\frac{1}{Z_1}\left(\frac{-2}{k_BT}\right)\exp[-\Delta/k_BT]$$

$$= 2N\gamma\left(\frac{\Delta}{V}\right)\frac{\exp[-\Delta/k_BT]}{1+2\exp[-\Delta/k_BT]}$$

$$= \underline{\left(\frac{\gamma}{V}\right)2N\Delta\,\frac{1}{\exp[\Delta/k_BT]+2}} = \gamma\,\frac{U}{V}$$

Problem 3 (30 points) Heating a Shell

a) For the shell,

$$P_{\text{in}} = 4\pi r^2 \sigma T_H^4$$

$$P_{\text{out}} = 4\pi R^2 \sigma T_S^4$$

$$P_{\text{out}} = P_{\text{in}} \Rightarrow r^2 T_H^4 = R^2 T_S^4 \rightarrow \underline{T_S = T_H \sqrt{\frac{r}{R}}}$$

b)

$$e(\omega, T)_{\text{heater}} = (1)\left(\frac{1}{4}\right) c\, u(\omega, T_H)$$

$$P_{\text{in}} = (4\pi r^2)\left(\frac{c}{4}\right)\int_0^{\omega_0}\left(\frac{k_B T_H}{\pi^2 c^3}\right)\omega^2\, d\omega$$

$$= \frac{r^2 k_B T_H}{\pi c^2}\int_0^{\omega_0}\omega^2\, d\omega = \underline{\frac{k_B \omega_0^3}{3\pi c^2}\, r^2 T_H}$$

Note that the power is coming from the central object (not from the shell) and from its surface (not volume). Thus this result is proportional to r^2.

c)

$$e(\omega, T)_{\text{shell}} = \alpha(\omega)\left(\frac{1}{4}\right) c\, u(\omega, T_S)$$

$$P_{\text{out}} = (4\pi R^2)\left(\frac{c}{4}\right)\int_0^{\omega_0}\left(\frac{k_B T_S}{\pi^2 c^3}\right)\omega^2\, d\omega = \frac{k_B \omega_0^3}{3\pi c^2}\, R^2 T_S$$

$$P_{\text{out}} = P_{\text{in}} \Rightarrow \underline{T_S = T_H\left(\frac{r}{R}\right)^2}$$

This is an example of a poor absorber being a poor emitter (Kirchoff's law, on the information sheet). The shell does not absorb beyond ω_0, thus it does not radiate beyond ω_0.