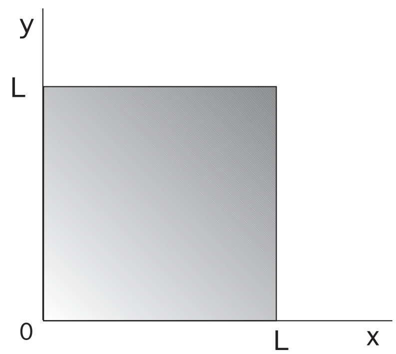
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Exam #1

Problem 1 (30 points) Quantum Dots



A complicated process creates quantum dots (also called artificial atoms) on the surface of a square chip of silicon. The probability density for finding a given dot at the location (x, y) on the chip is given by

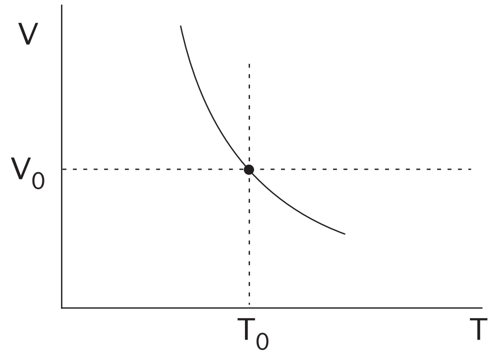
$$p(x,y) = \frac{1}{L^3}(x+y) \qquad \text{when } 0 \leq x \leq L \text{ and } 0 \leq y \leq L$$

$$= 0 \qquad \text{elsewhere}$$

a) (15 points) Find the conditional probability density for x given y, $p(x|y)$. Make a careful sketch of the result for the case $y = L/2$. Are x and y statistically independent random variables? Explain why or why not.

b) (15 points) [Note: you do not need the results of a) to do this part.] Find the probability density for the function $M \equiv \text{Max}(x, y)$ which takes on the value of the larger of the two quantities x and y. Sketch the result.

Problem 2 (30 points) A Real Gas



A theoretical model for a certain real (non-ideal) gas gives the following expressions for the internal energy and the pressure. a and b are constant parameters which, for simplicity, include the dependence of U and P on the total number of atoms.

$$U(T,V) = aV^{-2/3} + bV^{2/3}T^2$$

$$P(T,V) = (2/3)aV^{-5/3} + (2/3)bV^{-1/3}T^2$$

Find an expression $V = V(T, T_0, V_0)$ for the adiabatic path that passes through the point (T_0, V_0) in the V, T plane.

Problem 3 (40 points) Ultra-relativistic Gas in One Dimension

The microscopic state of each of the N identical atoms in a classical, weakly interacting, ultra-relativistic one dimensional gas can be described by three numbers: the atom's location x $(0 \leq x \leq L)$, the magnitude of its momentum p $(0 \leq p \leq \infty)$, and the sign of the momentum s $(s = +1$ or $-1)$. The energy associated with an individual atom is $\epsilon = cp$ where c is the velocity of light, and the total energy of the system is then

$$E = \sum_{i=1}^{N} \epsilon_i = c \sum_{i=1}^{N} p_i$$

Use the microcanonical ensemble with a total energy between E and $E + \Delta$ where $\Delta \ll E$ to find the properties of this gas.

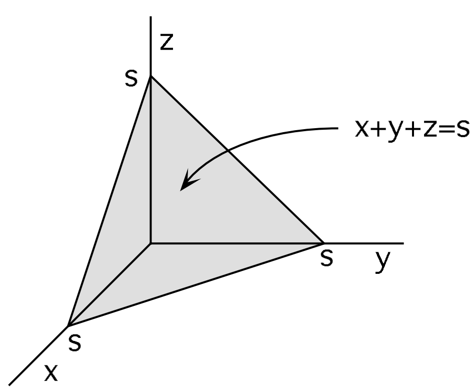
a) (16 points) Find the volume of the accessible region in phase space, $\Omega(E, L, N)$. Use the constant $\hbar$ which has the units of length times momentum to render Ω dimensionless, and make any adjustment necessary to assure that the Gibbs paradox is avoided. [You may want to make use of the mathematical result at the bottom of this page.]

b) (8 points) Find the energy relation, $E(T, L, N)$. Note that the term $\ln(N\Delta/E)$ can be neglected compared to other terms when computing the entropy of the gas.

c) (8 points) Find the tension $\mathcal{F}(T, L, N)$ in the gas. Note that $-\mathcal{F}$ can be viewed as the analogue of the pressure in the one-dimensional gas, that is, a negative value of $\mathcal{F}$ indicates that the gas is pushing out against its walls.

d) (8 points) Make use of the calculations you have already done above to find the equation of an adiabatic path $L = L(T, T_0, L_0)$ going through the point (T_0, L_0) in the T, L plane.

The following mathematical fact may be useful. The volume of a right angled pyramid of side S in 3 dimensions is $(1/6)S^3$. The volume of a right angled pyramid of side S in d dimensions is $(1/d!)S^d$.



PARTIAL DERIVATIVE RELATIONSHIPS

Let x, y, z be quantities satisfying a functional relation $f(x, y, z) = 0$. Let w be a function of any two of x, y, z. Then

$$\left(\frac{\partial x}{\partial y}\right)_w \left(\frac{\partial y}{\partial z}\right)_w = \left(\frac{\partial x}{\partial z}\right)_w$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

COMBINATORIAL FACTS

There are $K!$ different orderings of K objects. The number of ways of choosing L objects from a set of K objects is

$$\frac{K!}{(K-L)!}$$

if the order in which they are chosen matters, and

$$\frac{K!}{L!(K-L)!}$$

if order does not matter.

STERLING'S APPROXIMATION

When $K \gg 1$

$$\ln K! \approx K \ln K - K \quad \text{or} \quad K! \approx (K/e)^K$$

DERIVATIVE OF A LOG

$$\frac{d}{dx} \ln u(x) = \frac{1}{u(x)} \frac{du(x)}{dx}$$

VOLUME OF AN α DIMENSIONAL SPHERE OF RADIUS R

$$\frac{\pi^{\alpha/2}}{(\alpha/2)!} R^\alpha$$

LIMITS

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$$

$$\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

WORK IN SIMPLE SYSTEMS

System	Intensive quantity	Extensive quantity	Work
Hydrostatic system	P	V	$-PdV$
Wire	$\mathcal{F}$	L	$\mathcal{F}dL$
Surface	$\mathcal{S}$	A	$\mathcal{S}dA$
Reversible cell	E	Z	EdZ
Dielectric material	$\mathcal{E}$	$\mathcal{P}$	$\mathcal{E}d\mathcal{P}$
Magnetic material	H	M	HdM

INTEGRALS

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2\,e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int \frac{dx}{1+e^x} = \ln\left[\frac{e^x}{1+e^x}\right]$$

DEFINITE INTEGRALS

For integer n and m

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \sqrt{\pi}$$

$$(2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2\sigma^2}\,dx = 1\cdot 3\cdot 5\cdots(2n-1)\,\sigma^n$$

$$\int_0^\infty x\,e^{-x^2}\,dx = \frac{1}{2}$$

$$\int_0^1 x^m (1-x)^n\,dx = \frac{n!m!}{(m+n+1)!}$$

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