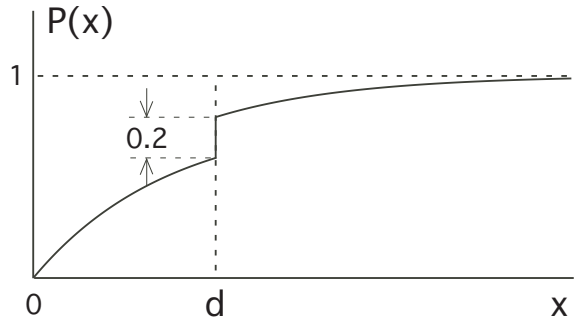


Solutions to Problem Set #1

Problem 1: Doping a Semiconductor

a) Mentally integrate the function $p(x)$ given in the figure. The result rises from zero at a decreasing rate, jumps discontinuously by 0.2 at $x = d$, then continues to rise asymptotically toward the value 1. This behavior is sketched below.



b)

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x p(x) dx = \frac{0.8}{l} \underbrace{\int_0^{\infty} x \exp(-x/l) dx}_{l^2} + 0.2 \underbrace{\int_0^{\infty} x \delta(x-d) dx}_d \\ &= \underline{0.8l + 0.2d} \end{aligned}$$

c)

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 p(x) dx = \frac{0.8}{l} \underbrace{\int_0^{\infty} x^2 \exp(-x/l) dx}_{2l^3} + 0.2 \underbrace{\int_0^{\infty} x^2 \delta(x-d) dx}_{d^2} \\ &= 1.6l^2 + 0.2d^2 \end{aligned}$$

$$\begin{aligned} \text{Var}(x) &\equiv \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \\ &= (1.6l^2 + 0.2d^2) - (0.64l^2 + 0.32ld + 0.04d^2) \\ &= \underline{0.96l^2 - 0.32ld - 0.16d^2} \end{aligned}$$

d)

$$\begin{aligned}\langle \exp(-x/s) \rangle &= \int_{-\infty}^{\infty} \exp(-x/s) p(x) dx \\ &= \frac{0.8}{l} \underbrace{\int_0^{\infty} \exp(-x/s) \exp(-x/l) dx}_{(1/s + 1/l)^{-1}} + 0.2 \underbrace{\int_0^{\infty} \exp(-x/s) \delta(x - d) dx}_{\exp(-d/s)} \\ &= \underline{\left(\frac{0.8}{1 + l/s} \right) + 0.2 \exp(-d/s)}\end{aligned}$$

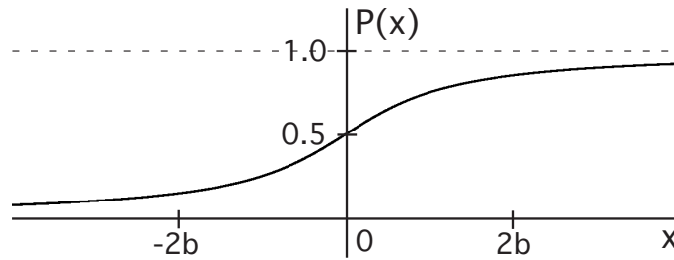
Check to see that this result is physically reasonable. Note that if the skin depth s is much less than the distance d , the impurities on the grain boundary do not contribute to the surface impedance. Similarly, if the skin depth is much less than the characteristic diffusion distance l , the impurity contribution to the surface impedance is greatly reduced.

Problem 2: A Peculiar Probability Density

$$\begin{aligned}
 \text{a)} \quad 1 &= \int_{-\infty}^{\infty} p(x) dx = 2 \int_0^{\infty} \frac{a}{b^2 + x^2} dx \\
 &= \frac{2a}{b} \underbrace{\int_0^{\infty} \frac{1}{1 + \xi^2} d\xi}_{\pi/2} = (\pi a/b)
 \end{aligned}$$

$$a = \underline{(b/\pi)}$$

$$\begin{aligned}
 \text{b)} \quad P(x) &= \int_{-\infty}^x p(x') dx' = \frac{b}{\pi} \int_{-\infty}^x \frac{1}{b^2 + x'^2} dx' \\
 &= \frac{b}{\pi} \left[\frac{1}{b} \arctan(x'/b) \right]_{-\infty}^x \\
 &= \underline{\frac{1}{\pi} \arctan(x/b) + \frac{1}{2}}
 \end{aligned}$$



c) $\underline{\langle x \rangle = 0}$ by symmetry. $p(x)$ is an even function and x is odd.

d) $p(x)$ falls to half its value at $\underline{x = \pm b}$.

e)

$$\langle x^2 \rangle = \frac{b}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{b^2 + x^2} dx$$

However the limit of $x^2/(b^2 + x^2)$ as $x \rightarrow \pm \infty$ is unity, so this integral diverges. Neither the mean square nor the Variance of this distribution exist.

Problem 3: Visualizing the Probability Density for a Classical Harmonic Oscillator

a) First find the velocity as a function of time by taking the derivative of the displacement with respect to time.

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt}[x_0 \sin(\omega t + \phi)] \\ &= \omega x_0 \cos(\omega t + \phi)\end{aligned}$$

But we don't want the velocity as a function of t , we want it as a function of the position x . And, we don't actually need the velocity itself, we want the speed (the magnitude of the velocity). Because of this we do not have to worry about losing the sign of the velocity when we work with its square.

$$\begin{aligned}\dot{x}^2(t) &= (\omega x_0)^2 \cos^2(\omega t + \phi) \\ &= (\omega x_0)^2 [1 - \sin^2(\omega t + \phi)] \\ &= (\omega x_0)^2 [1 - (x(t)/x_0)^2]\end{aligned}$$

Finally, the speed is computed as the square root of the square of the velocity.

$$\underline{|\dot{x}(t)|} = \omega(x_0^2 - x^2(t))^{1/2} \quad \text{for } |x(t)| \leq x_0$$

b) We are told that the probability density for finding an oscillator at x is proportional to the the time a given oscillator spends near x , and that this time is inversely proportional to its speed at that point. Expressed mathematically this becomes

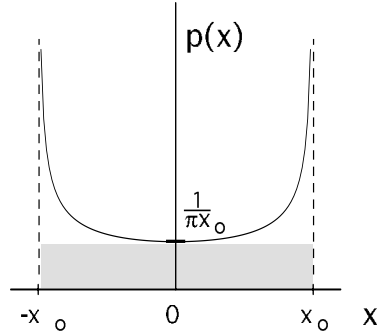
$$\begin{aligned}p(x) &\propto |\dot{x}(t)|^{-1} \\ &= C(x_0^2 - x^2)^{-1/2} \quad \text{for } |x| < x_0\end{aligned}$$

where C is a proportionality constant which we can find by normalizing $p(x)$.

$$\begin{aligned}\int_{-\infty}^{\infty} p(x) dx &= C \int_{-x_0}^{x_0} (x_0^2 - x^2)^{-1/2} dx \\ &= 2C \int_0^{x_0} \frac{dx/x_0}{\sqrt{1 - (x/x_0)^2}} \quad \text{let } x/x_0 \equiv y \\ &= 2C \underbrace{\int_0^1 \frac{dy}{\sqrt{1 - y^2}}}_{\pi/2} \\ &= \pi C \\ &= 1 \quad \text{by normalization}\end{aligned}$$

The last two lines imply that $C = 1/\pi$. We can now write (and plot) the final result.

$$\begin{aligned}
 p(x) &= \left(\pi x_0 \sqrt{1 - (x/x_0)^2} \right)^{-1} & |x| < x_0 \\
 &= 0 & |x| > x_0
 \end{aligned}$$

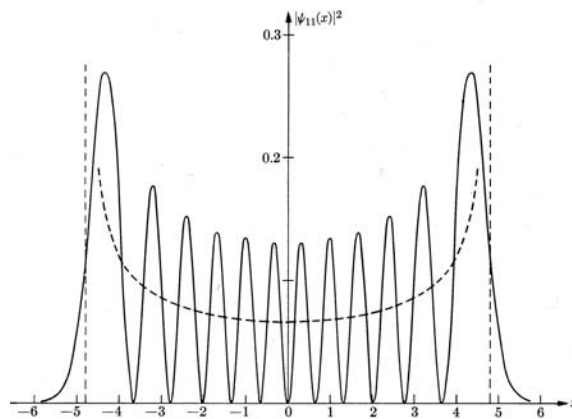


As a check of the result, note that the area of the shaded rectangle is equal to $2/\pi$. The area is dimensionless, as it should be, and is a reasonable fraction of the anticipated total area under $p(x)$, that is 1.

c) The sketch of $p(x)$ is shown above. By inspection the most probable value of x is $\pm x_0$ and the least probable accessible value of x is zero. The mean value of x is zero by symmetry. It is the divergence of $p(x)$ at the turning points that gives rise to the apparent image of the pencil at these points in your experiment.

COMMENTS If an oscillator oscillates back and forth with some fixed frequency, why is this $p(x)$ independent of time? The reason is that we did not know the starting time (or equivalently the phase ϕ) so we used an approach which effectively averaged over all possible starting times. This washed out the time dependence and left a time-independent probability. If we had known the phase, or equivalently the position and velocity at some given time, then the process would have been deterministic. In that case $p(x)$ would be a delta function centered at a value of x which oscillated back and forth between $-x_0$ and $+x_0$.

Those of you who have already had a course in quantum mechanics may want to compare the classical result you found above with the result for a quantum harmonic oscillator in an energy eigenstate with a high value of the quantum number n and the same total energy. Will this probability be time dependent? No. Recall why the energy eigenstates of a potential are also called “stationary states”.



Problem 4: Quantized Angular Momentum

a) Using the expression for the normalization of a probability density, along with expressions for the mean and the mean square, we can write three separate equations relating the individual probabilities.

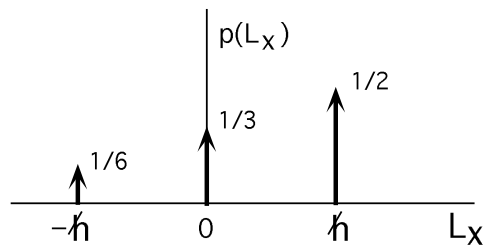
$$\begin{aligned} p(-\hbar) + p(0) + p(\hbar) &= 1 \\ -\hbar p(-\hbar) + 0 \times p(0) + \hbar p(\hbar) &= \langle L_x \rangle = \frac{1}{3}\hbar \\ \hbar^2 p(-\hbar) + 0 \times p(0) + \hbar^2 p(\hbar) &= \langle L_x^2 \rangle = \frac{2}{3}\hbar^2 \end{aligned}$$

We now have three simple linear equations in three unknowns. The last two can be simplified and solved for two of our unknowns.

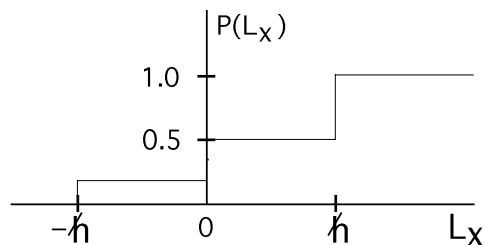
$$\left. \begin{aligned} -p(-\hbar) + p(\hbar) &= \frac{1}{3} \\ p(-\hbar) + p(\hbar) &= \frac{2}{3} \end{aligned} \right\} \Rightarrow \begin{aligned} p(\hbar) &= \frac{1}{2} \\ p(-\hbar) &= \frac{1}{6} \end{aligned}$$

Substitute these results into the first equation to find the last unknown.

$$\frac{1}{6} + p(0) + \frac{1}{2} = 1 \Rightarrow \underline{p(0) = \frac{1}{3}}$$



b)



Problem 5: A Coherent State of a Quantum Harmonic Oscillator

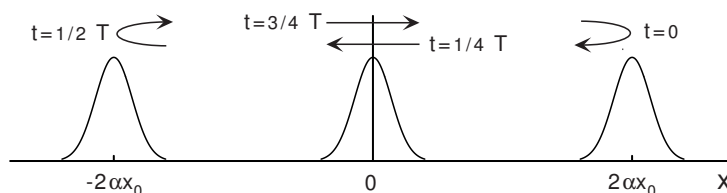
$$\Psi(\vec{r}, t) = (2\pi x_0^2)^{-1/4} \exp \left[-\frac{i\omega t}{2} - \frac{i}{(2x_0)^2} (2\alpha x x_0 \sin \omega t - \alpha^2 x_0^2 \sin 2\omega t) - \left(\frac{x - 2\alpha x_0 \cos \omega t}{2x_0} \right)^2 \right]$$

a) First note that the given wavefunction has the form $\Psi = a \exp[ib + c] = a \exp[ib] \exp[c]$ where a , b and c are real. Thus the square of the magnitude of the wavefunction is simply $a^2 \exp[2c]$ and finding the probability density is not algebraically difficult.

$$p(x, t) = |\Psi(x, t)|^2 = \frac{1}{\sqrt{2\pi x_0^2}} \exp \left[-\frac{(x - 2\alpha x_0 \cos \omega t)^2}{2x_0^2} \right]$$

b) By inspection we see that this is a Gaussian with a time dependent mean $\langle x \rangle = 2\alpha x_0 \cos \omega t$ and a time independent standard deviation $\sigma = x_0$.

c) $p(x, t)$ involves a time independent pulse shape, a Gaussian, whose center oscillates harmonically between $-2\alpha x_0$ and $2\alpha x_0$ with radian frequency ω .



Those already familiar with quantum mechanics will recognize this as a “coherent state” of the harmonic oscillator, a state whose behavior is closest to the classical behavior. It is **not** an energy eigenstate since $p(x)$ depends on t . It should be compared with a classical harmonic oscillator with known phase ϕ and the same maximum excursion: $x = 2\alpha x_0 \cos \omega t$. In this deterministic classical case $p(x, t)$ is given by

$$p(x, t) = \delta(x - 2\alpha x_0 \cos \omega t).$$

The coherent state is a good representation of the quantum behavior of the electromagnetic field of a laser well above the threshold for oscillation.

Problem 6: Bose-Einstein Statistics

We are given the discrete probability density

$$p(n) = (1 - a)a^n \quad n = 0, 1, 2, \dots$$

a) First we find the mean of n .

$$\langle n \rangle = \sum_{n=0}^{\infty} np(n) = (1 - a) \underbrace{\sum_{n=0}^{\infty} na^n}_{S_1}$$

The sum S_1 can be found by manipulating the normalization sum.

$$\sum_{n=0}^{\infty} p(n) = \sum_{n=0}^{\infty} (1 - a)a^n = (1 - a) \sum_{n=0}^{\infty} a^n \text{ must} = 1$$

Rearranging the last two terms gives the sum of a geometric series:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1 - a}.$$

But note what happens when we take the derivative of this result with respect to the parameter a .

$$\begin{aligned} \frac{d}{da} \sum_{n=0}^{\infty} a^n &= \sum_{n=0}^{\infty} na^{n-1} = \frac{1}{a} \sum_{n=0}^{\infty} na^n = \frac{1}{a} S_1 \\ \text{also} &= \frac{d}{da} \left(\frac{1}{1 - a} \right) = \frac{1}{(1 - a)^2} \end{aligned}$$

Equating the two results gives the value of the sum we need, $S_1 = a/(1 - a)^2$, and allows us to finish the computation of the mean of n :

$$\underline{\langle n \rangle = \frac{a}{1 - a}}.$$

c) To find the variance we first need the mean of the square of n .

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 p(n) = (1 - a) \underbrace{\sum_{n=0}^{\infty} n^2 a^n}_{S_2}$$

Now try the same trick used above, but on the sum S_1 .

$$\frac{d}{da} \underbrace{\sum_{n=0}^{\infty} na^n}_{S_1} = \sum_{n=0}^{\infty} n^2 a^{n-1} = \frac{1}{a} \sum_{n=0}^{\infty} n^2 a^n = \frac{1}{a} S_2$$

$$\text{also} = \frac{d}{da} \left(\frac{a}{(1-a)^2} \right) = \frac{2a}{(1-a)^3} + \frac{1}{(1-a)^2}$$

Then

$$\langle n^2 \rangle = (1-a) \left[\frac{2a^2}{(1-a)^3} + \frac{a}{(1-a)^2} \right] = 2 \left(\frac{a}{1-a} \right)^2 + \left(\frac{a}{1-a} \right)$$

$$= 2 \langle n \rangle^2 + \langle n \rangle,$$

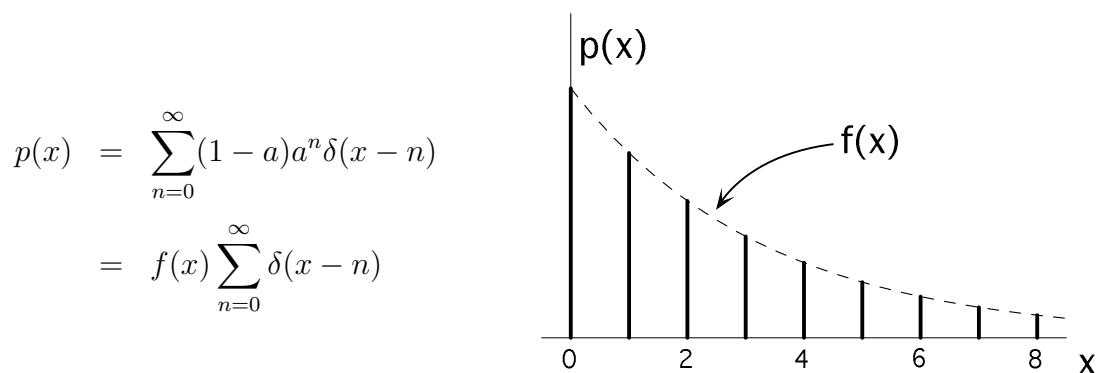
and

$$\text{Variance} = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle^2 + \langle n \rangle - \langle n \rangle^2$$

$$= \underline{\langle n \rangle (1 + \langle n \rangle)}.$$

This is greater than the variance for a Poisson, $\langle n \rangle$, by a factor $\underline{1 + \langle n \rangle}$.

c)



$$\text{Try } f(x) = C e^{-x/\phi},$$

$$\text{then } f(x=n) = C e^{-n/\phi} = C (e^{-1/\phi})^n = (1-a)a^n.$$

This tells us that $C = 1 - a$ and $\exp(-1/\phi) = a$. We can invert the expression found above for $\langle n \rangle$ to give a as a function of $\langle n \rangle$: $a = \langle n \rangle / (1 + \langle n \rangle)$.

$$-1/\phi = \ln a = \ln \left(\frac{\langle n \rangle}{1 + \langle n \rangle} \right)$$

$$1/\phi = \ln \left(\frac{\langle n \rangle + 1}{\langle n \rangle} \right) = \underline{\underline{\ln \left(1 + \frac{1}{\langle n \rangle} \right)}}$$

Recall that for small x one has the expansion $\ln(1+x) = x - x^2/2 + \dots$. Therefore in the limit $\langle n \rangle \gg 1$, $1/\phi \rightarrow 1/\langle n \rangle$ which implies $\phi \rightarrow \langle n \rangle$.

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