PROFESSOR: All right. Welcome, everyone. Hi. So today we're going to pick up where we left off last time in our study of angular momentum and rotations in quantum mechanics. Before I get started, let me open up for questions, pragmatic and physics related. Yeah?

AUDIENCE: When we were solving for the 3D harmonic oscillator we solved for the energy eigenfunction that was a product of 5x, 5y, and 5z. We made an assumption that 5e was equal to 5 sub n of x plus that sum. How did you get from [INAUDIBLE]?

PROFESSOR: Good. So what we had was that we had that the energy-- we wanted to find the energy of the 3D harmonic oscillator. And we wanted to find the energy eigenfunctions and eigenvalues. And they way we did this was by saying, look, the energy of the 3D harmonic oscillator, which I can think of as a function of x and px and y and py and z and pz, has this nice form. We could write it as the energy operator purely in terms of x, p squared x upon 2m plus m omega squared upon 2x squared. Plus-- so this is a single 1D harmonic oscillator energy operator in the x direction. Plus E 1D in the y direction, plus a harmonic oscillator energy 1D in the z direction. So that was the first observation.

And then we said that given that this splits in this fashion, I'm going to write my energy eigenfunction, phi of x, y, and z in separated form as a product. Phi x of x, phi y of y, and phi z of z. And we used this and deduced that in order for this to be an eigenfunction of the 3D harmonic oscillator it must be true that 5x of x was itself an eigenfunction of the x harmonic oscillator equals with some energy epsilon, which I will call epsilon sub x, phi sub x. And ditto for y and z.

And if this is true, if phi sub x is an eigenfunction of the 1D harmonic oscillator, then

this is an eigenfunction of the 3D harmonic oscillator. But we know what the eigenfunctions are of the harmonic oscillator. The eigenfunctions of the 1D harmonic oscillator we gave a name. We call them pi sub n.

So what we then said was, look, if this is an eigenfunction of the 1D harmonic oscillator in the x direction, then it's labeled by an n. And if this one is in the y direction, it's labeled by an l. And if this one is in the z direction it's labeled by a z. But on top of that we know more. We know what these energy eigenvalues are. The energy eigenvalues corresponding to these guys are if this is phi n, this is En. I can simply write this as En for the 1D harmonic oscillator.

And taking this, using the fact that they're 1D eigenfunctions and plugging it into the energy eigenvalue equation for the 3D harmonic oscillator tells us that the energy eigenvalues for the 3D case are of the form E1d x plus E1d in the y plus E1d in the z, which is equal to, since these were the same frequency, h bar omega times n plus l plus m, from each of them a 1/2, so plus 3/2. Cool? That answer your question? Excellent. Other questions? Yeah?

AUDIENCE: Energy and angular momentum have to be related somehow, right?

PROFESSOR: Yeah.

AUDIENCE: I mean, of course. Because it's both.

PROFESSOR: Indeed.

AUDIENCE: [INAUDIBLE]. The thing is, we have a ladder. Is there limits on them?

PROFESSOR: Very good question. So this is where we're going to pick up. Let me rephrase this. This is really two questions. Question number one. Look, there should be a relationship between angular momentum and energy. But we're just talking about angular momentum. Why? Second question, look, we've got a ladder. But is the ladder infinite? So let me come back to the second question. That's going to be the beginning of the lecture.

On the first question, yes angular momentum is going to play a role when we

calculate the energy. But two quick things to note. First off, consider a system which is spherically symmetric, rotationally invariant. That means that the energy doesn't depend on a rotation. If I rotate the system I haven't changed the energy. So if the system is rotationally invariant, that's going to imply some constraints on the energy eigenvalues and how they depend on the angular momentum, as we discussed last time.

Let me say that slightly differently. When we talk about the free particle, 1D free particle-- we've talked about this one to death. Take the 1D free particle. We can write the energy eigenfunctions as momentum eigenfunctions, because the momentum commutes with the energy. And so the way the eigenfunctions of the energy operate are indeed e to iKX there are plane waves, they're eigenfunctions of the momentum operator as well.

Similarly, when we talk about a 3D system it's going to be useful in talking about the energy eigenvalues to know a basis of eigenfunctions of the angular momentum operator. Knowing the angular momentum operator is going to allow us to write energy eigenfunctions in a natural way and a simply way, in the same way that knowing the momentum operator allowed us to write energy eigenfunctions in a simple way in the 1D case. That make sense? We're going to have a glorified version of the Fourier theorum where instead of something over e to iKX, we're going to have something over angular momentum eigenstates. And those are called the spherical harmonics. And they are the analog of Fourier expansion for this year.

But you're right. We're going to have to understand how that interacts with the energy. And that'll be the topic of the next lecture. We're going to finish up angular momentum today. Other questions?

OK, so from last time, these are the commutation relations which we partially derived in lecture and which you will be driving on your problem set. It's a really good exercise. Commit these to memory. They're your friends.

Key thing here to keep in mind. h bar has units of angular momentum, so this makes sense. Angular momentum, angular momentum, angular momentum. So

when you see an h bar in this setting, its job, in some sense, is to make everything dimensionally sensible.

So the important things here are that lx and ly do not commute. They commute to lz. Can you have a state with definite angular momentum in the x direction and definite angular momentum in the y direction simultaneously? No, because of this commutation relation. It would have to vanish.

This is, say, the x component of the angular momentum. Can you have a state with definite angular momentum in the x direction and total angular momentum all squared? Yes. OK, great. That's going to be important for us. So in order to construct the eigenfunctions it turns out to be useful to construct these so-called raising and lowering operators, which are Lx plus iLy. They have a couple of nice properties. The first is, since these are built up out of Lx and Ly, both of which commute with L squared, the L plus minuses commute with L squared.

So these guys commute. So if we have an eigenfunction of L squared, acting with L plus does not change its eigenvalue. Similarly, L plus commuting with Lz gives us h bar L plus or minus, with a plus or minus out front. This is just like the raising and lowering operators for the harmonic oscillator. But instead of the energy we have the angular momentum.

So this is going to tell us that the angular momentum eigenvalues, the eigenvalues of Lz, are shifted by plus or minus h bar when we raise or lower with L plus or L minus, just like the energy was shifted-- for the 1d harmonic oscillator the energy was shifted, plus h bar omega a dagger, was shifted by h bar omega when we acted with a plus on an energy eigenstate. Same thing.

Questions on the commutators before we get going? In some sense, we're going to just to just take advantage of these commutation relations and explore their consequences today. So our goal is going to be to build the eigenfunctions and eigenvalues of the angular momentum operators, and in particular of the most angular momentum operators we [INAUDIBLE] complete set of commuting observables, L squared and Lz.

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You might complain, look, why Lz? Whoops. I don't mean a commutator. I mean the set. You might say, why Lz? Why not Lx? And if you call this the z direction then I will simply choose a new basis where this is called the x direction. So it makes no difference whatsoever. It's just a name. The reason we're going to choose Lz is because that coordinate system plays nicely with spherical coordinates, just the conventional choice of spherical coordinates where theta equals 0 is the up axis. But there's nothing deep about that. We could have taken any of these.

OK. So this is our goal. So let's get started. So first, because of these commutation relations and in particular this one, we know that we can find common eigenfunctions of L squared and Lz. Let us call those common eigenfunctions by a name, Y sub lm such that-- so let these guys be the common eigenfunctions of L squared and Lz. I.e., L squared Ylm is equal to-- well first off, units. This has units of angular momentum squared, h bar squared. So that got rid of the units. And we want our eigenvalue, lm. And because I know the answer, I'm going to give-- instead of calling this a random dimensionless number, which would be the eigenvalue, I'm going to call it a very specific thing, l, l plus 1. This is a slightly grotesque thing to do, but it will make the algebra much easier.

So similarly, so that's what the little l is. Little l is labeling the eigenvalue of L squared and the actual value of that eigenvalue. I'm just calling h bar squared ll plus 1. That doesn't tell you anything interesting. This was already a real positive number. So this could have been any real positive number as well, by tuning L.

Similarly, Lz Ylm-- I want this to be an eigenfunction-- this has units of angular momentum. So I'll put an h bar. Now we have a dimensionless coefficient Ylm. And I'll simply call that m. So if you will, these are the definitions of the symbols m and little l. And Ylm are just the names I'm giving to the angular momentum eigenfunctions. Cool? So I haven't actually done anything. I've just told you that these are the eigenfunctions. What we want to know is what properties do they have, and what are the actual allowed values of the eigenvalues.

So the two key things to note are first that L plus and minus leave the eigenvalue of

L squared alone. So they leave l alone. So this is the statement that L squared on L plus Ylm is equal to L plus on L squared Ylm is equal to h bar squared ll plus 1, the eigenvalue of L squared acting on Ylm, L plus Ylm. And so L plus Ylm is just as much an eigenfunction of L squared as Ylm was itself, with the same eigenvalue.

Two. So that came from-- where are we--- this commutation relation. OK, so similarly, L plus minus raise or lower m by one. And the way to see that is to do exactly the same computation, Lz on L plus, for example, Ylm is equal to L plus-- now we can write the commutator-- Lz, L plus, plus L plus Lz Ylm. But the commutator of Lz with L plus we already have, is plus h bar L plus. So this is equal to h bar L plus.

And from this term, L plus Lz of Ylm, Lz acting on Ylm gives us h bar m, plus h bar m L plus Ylm is equal to, pulling this out, this is h bar times m plus 1 times L plus. h bar m plus 1 L plus Ylm. So L plus has raised the eigenvalue m by one. This state, what we get by acting on Ylm with the raising operator is a thing with m greater by one. And that came from the commutation relation. And if we had done the same thing with minus, if you go through the minus signs, it just gives us this.

What does that tell us? What this tells us is we get a ladder of states. Let's look at what they look like. Each ladder, for a given value of L, if you raise with L plus and lower with L minus, you don't change L but you do change m. So we get ladders that are labeled by L. So for example, if I have some value L1, this is going to give me some state labeled by m. Let me put the m to the side. I can raise it to get m plus 1 by L plus. And I can lower it to get m minus 1 by L minus.

So I got a tower. And if we have another value, a different value of L, I'll call it L2, we got another tower. You get m, m plus 1, m plus 2, dot, dot, dot, minus 1, dot, dot, dot. So we have separated towers with different values of L squared. And within each tower we can raise and lower by L plus, skipping by one. OK, questions?

So that was basically the end of the last lecture, said slightly differently. Now here's the question. So this is the question that a student asked right at the beginning. Is

this tower infinite? Or does it end? So I pose to you the question. Is the tower infinite, or does it end? And why?

AUDIENCE: [INAUDIBLE] direction.

PROFESSOR: OK, so it's tempting to say it's infinite in one direction, because--?

AUDIENCE: There are no bounds to the angular momentum.

PROFESSOR: OK, so it's because there are no bounds to the angular momentum one can have. That's tempting.

AUDIENCE: But at the same time, when you act a raising or lowering operator on L, the eigenvalue of L squared remains the same. So then you can't raise the z [INAUDIBLE] of the angular momentum above the actual momentum.

PROFESSOR: Thank you. Exactly. So here's the statement. Let me restate that. That's exactly right. So look, L squared is the eigenvalue of the total angular momentum. Roughly speaking, it's giving you precisely in the state Ylm it tells you the expected value of the total angular momentum, L squared. That's some number.

Now, if you act with L plus you keep increasing the expected value of Lz. But if you keep increasing it and keep increasing it and keep increasing it, Lz will eventually get much larger than the square root of L squared. That probably isn't true. That sounds wrong. That was the statement. Excellent. Exactly right. So let's make that precise.

So is the tower infinite? No. It's probably not, for precisely that reason. So let's make that precise. So here's the way we're going to do it. This is a useful trick in general. This will outlive angular momentum and be a useful trick throughout quantum mechanics for you. I used it in a paper once.

So here's the nice observation. Suppose it's true that the tower ends. Just like for the raising and lowering operators for the harmonic oscillator in one dimension, that tells us that in order for the power to end that state must be 0 once we raise it. The last state, so Yl, and I'll call this m plus, must be 0. There must be a max.

Oh, sorry. Let me actually, before I walk through exactly this statement-- So let me make, first, this, the no, slightly more obvious and precise. So let's turn that argument into a precise statement. L squared is equal to, just from the definition, L_x squared plus L_y squared plus L_z squared. Now let's take the expectation value in this state Y_{lm} of both sides of this equation. So on the left-hand side we get h bar squared $l(l+1)$. And on the right-hand side we get the expectation value of L_x squared plus the expectation value of L_y squared plus the expectation value of L_z squared. But we know that the expectation value of L_z squared is h bar squared, m squared.

But the expectation of L_x squared and L_y squared are strictly positive. Because this can be written as a sum over all possible eigenvalues of L_x squared, which is the square of the possible eigenvalues of L_x times the probability distribution. That's a sum of positive, strictly positive [INAUDIBLE]. These are positive [INAUDIBLE].

So $l(l+1)$ is equal to positive plus positive plus h bar squared m squared. In particular that tells you that it's greater than or equal to h bar squared m squared. Maybe these are 0. So the least it can be is-- so the most m squared can possibly be is the square root of L plus 1. Or the most m can be. So m is bounded. There must be a maximum m, and there must be a minimum m. Because this is squared. The sign doesn't matter. Everyone cool with that?

OK, so let's turn this into, now, a precise argument. What are the values of m plus and m minus? What is the top of the value of each tower? Probably it's going to depend on total L, right? So it's going to depend on each value of L. Let's check that.

So here's the nice trick. Suppose we really do have a maximum m plus. That means that if I try to raise the state Y_{lm} plus, I should get the state 0, which ends the tower. So suppose this is true. In that case, in particular here's the nice trick, L plus Y_{lm}, if we take its magnitude, if we take the magnitude of this state, the state is 0. It's the zero function. So what's its magnitude? Zero. You might not think that's all that impressive an observation.

OK, but note what this is. We know how to work with this. And in particular this is equal to-- I should have done this, dot, dot, dot-- I'm now going to use the Hermitian adjoint and pull this over to the right-hand side. The adjoint of L plus is L minus. So this gives us Ylm, L minus, L plus Ylm. This also doesn't look like much of an improvement, until you notice from the definition of L plus and L minus that what's L minus L plus? Well, L minus L plus, we're going to get an Lx squared. So Ylm, we get an Lx squared plus an Ly squared. I'm going to get my sign right, if I'm not careful. Plus-- well, let's just do it.

So we have L minus L plus. So we're going to get a Lx iLy minus i, iLyLx. So plus i commutator of Lx with Ly. Good. So that's progress, Ylm. But it's still not progress, because the natural operators with which to act on the Ylm's are L squared and Lz. So can we put this in the form L squared and Lz? Sure. This is L squared minus Lz squared. Ylm L squared minus Lz squared. And this is i h bar Lz times i is minus h bar Lz. So minus h bar Lz, Ylm. And this must be equal to 0.

But this is equal to-- from L squared we get h bar squared, ll plus 1. From the Lz squared we get minus m squared h bar squared. And from here we get minus h bar, h bar m, so minus m each bar squared. Notice that the units worked out. And all of this was multiplying Ylm, Ylm. But Ylm, Ylm, if it's a properly normalized eigenstate, which is what we were assuming at the beginning, is just 1. So we get this times 1 is 0. Aha.

And notice that in all of this, this was m plus. We were assuming, we were working here with the assumption that L plus annihilated this top state. And so grouping this together, this says therefore h bar squared times-- pulling out a common h bar squared from of all this-- times l, l plus 1 minus m plus, m plus, plus 1 is equal to 0. And this tells us that m plus is equal to l. And if you want to be strict, put a plus sign. Everyone cool with that? This is a very useful trick. If you know something is 0 as function, you know its norm is 0. And now you can use things like Hermitian adjoints. Very, very useful. OK questions about that? Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Excellent. Where this came from is I was just literally taking L plus and L minus, taking the definitions, and plugging them in. So if I have L minus L plus, L minus is, just to write it out explicitly, L minus is equal to Lx. Because the [INAUDIBLE] are observables, and so they're Hermitians, so they're self-adjoint, the minus iLy. So L minus L plus gives me an LxLx. It gives me an LxiLy. It gives you a minus iLyLx. And it gives me a minus ii, which is plus 1 Ly squared.

Cool? Excellent. I'm in a state of serious desperation here. Good. Other questions? Yeah.

AUDIENCE: [INAUDIBLE] How do you get from that to that?

PROFESSOR: This to here? OK, good. Sorry, I jumped a step here. So here what I said is, look, I've got this nice set of operators acting on Ylm. I know how each of these operators acts on Ylm. Lz gives me an h bar m, minus some h bar squared. Lz squared, L squared. And then overall that was a just some number times Ylm, which I can pull out of the inner product. Cool? Yeah?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Good. This subscript plus meant that, look, there was a maximum value of m. So m squared had to be less than or equal to this. There's a maximum value and a minimum value. Maybe they're different. I don't know. I'm just going to be open-minded about that. Others? Yeah?

AUDIENCE: Will that mean, then, that you can't ever have all the angular momentum [INAUDIBLE]?

PROFESSOR: Yeah, awesome observation. Exactly. We were going to get there in a little bit. I like that. That's exactly right. Let me go through a couple more steps, and then I'll come back to that observation. And I promise I will say so.

So if we go through exactly a similar argument, let's see what happens if we did L minus on Ylm minus, just to walk through the logic. So if this were-- you lower the

lowest one, you get 0. Then we'd get the same story, L minus L minus. When we take the adjoint we'd get L plus, L minus. And L plus L minus, what changes? The only thing that's going to change is that you get this commutator the other direction. And there are minus signs in various places. The upshot of which is that m minus is equal to minus l.

So this is quite parsimonious. It's symmetric. If you take z to minus z, if you switch the sign of the angular momentum you get the same thing back. That's satisfying, perhaps. But it's way more than that. This tells us a lot about the possible eigenvalues, in the following way.

Look back at our towers. In our towers we have that the angular momentum is raised and lowered by L plus. So the Lz angular momentum is raised and lowered by L plus. l remains the same. But there's a maximum state now, which is plus l. And there's a minimum stay in here, l1. And here there's a minimum state, minus l1. Similarly for the other tower, there's a minimum state minus l2 and a maximum state, l2.

So how big each tower is depends on the total angular momentum. That kind of makes sense, right? If you've got more angular momentum and you can only step Lz by one, you've got more room to move with a large value of l than with a small value of l. OK.

So what does that tell us about the values of m? Notice that m spans the values from its minimal value, minus l to l in integer steps, in unit steps. So if you think about m is l, then m is l minus 1. And if I keep lowering I get down to m is minus l plus 1. And then m is equal to minus l.

So the difference in the Lz eigenvalue between these guys, the difference is 2l. But the number of unit steps in here is one fewer, because one, dot, dot, dot 2l minus 1. So this is some integer, which is the number of states minus 1. So if these are n states, and I'll call this N sub l states, then this difference twice l is N sub l minus 1, because it's unit steps. Cool?

So for example, if there were two states, so N is 2, 2l is equal to-- well, that's 1, which is 2 minus 1, the number of states minus 1, also known as 1. So l is 1/2. And more generally we find that l must be, in order for this process to make sense, in order for the m plus and the m minus to match up, we need that l is of the form an integer N sub l minus 1 upon 2.

So this tells us that where Nl is an integer-- Nl is just the number of states in this tower, and it's a strictly positive integer. If you have zero states in the tower, then that's not very interesting. So this tells us that l is an integer a half integer, but nothing else. Cool? In particular, if it's a half integer, that means 1/2, 3/2, 5/2, if it's an integer or it can be 0. There's nothing wrong with it being 0. But then 1 on.

So that means we can plot our system in the following way. If we have l equals 0, how many states do we have? If little l is 0, how many states are? What's the largest value of Lz? What's the largest allowed value of Lz or of m if little l is equal to 0?

AUDIENCE: 0.

PROFESSOR: 0. Because it goes from plus 0 to minus 0. So that's pretty much 0. So we have a single state with m equals 0 and l equals 0. If l is equal to-- what's the next possible value?-- 1/2, then m can be either 1/2 or we lower it by 1, which is minus 1/2. So there are two states, m equals 1/2 and m equals minus 1/2. So there's one power. It's a very short tower. This is the shortest possible tower.

Then we also have the state l equals 1, which has m equals 0. It has m equals 1. And it has m equals minus 1. And that's it. And so on and so forth. Four states for l equals 3/2, with states 3/2, 1/2, this is m equals 1/2, m equals minus 1/2, and minus 3/2. The values of m span from minus l to l with integer steps.

And this is possible for all values of l which are of the form all integer or half integer l's. So for every different value of the total angular momentum, the total amount of angular momentum you have, you have a tower of states labeled by Lz's eigenvalue in this fashion. Questions?

Does anyone notice anything troubling about these? Something physically a little

discomfiting about any of these? What does it mean to say I'm in the state l equals 0, m equals 0? What is that telling you? Zero angular momentum. What's the expectation value of l squared? Zero. The expectation value of Lz? And by rotational invariance, the expected value of Lx or Ly? That thing is not rotating. There's no angular momentum. So the no angular momentum state is one with l equals 0, m equals 0. So this is not spinning.

What about this guy? L equals 1, m equals 0. Does this thing carry angular momentum? Yeah, absolutely. So it just doesn't carry any angular momentum in the z direction. But its total angular momentum on average, its expected total angular momentum is ll plus 1, which is 2, times h bar squared. So if you measure Lx and Ly what do you expect to get? Well, if you measure Lx squared or Ly squared you probably expect to get something non-zero. We'll come back to that in just a second.

And what about the state l equals 1, m equals 1? Your angular momentum-- you've got as much angular momentum in the z direction as you possibly can. So that definitely carries angular momentum. So there's a state that has no angular momentum in the z direction. And there's a state that has some, and there's a state that has less. That make sense. Yeah?

AUDIENCE: If m equals 1 and l equals 1, that means that the angular momentum for the z direction is the total angular momentum?

PROFESSOR: Is it?

AUDIENCE: And then Lx and Ly was 0?

PROFESSOR: OK, that's an excellent question. Let me answer that now, and then I'll come back to the point I wanted to make. Hold on a second. Let me just answer that question. I'll work here.

So here's the crucial thing. Even in this state, so you were asking about the state l equals 1, m equals 1. And the question that was asked, a very good question, is look, does that mean all the angular momentum is in the z direction, and Lx and Ly

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are 0? But let me just ask this more broadly. Suppose we have a state with angular momentum l, and m is equal to l. The most angular momentum you can possibly have in the z direction, same question. Is all the angular momentum in the z direction?

And what I want to emphasize you is no, that's absolutely not the case. So two arguments for that. The first is, suppose it's true that Lx and Ly are identically 0. Can that satisfy the uncertainty relation due to those commutators? No. There must be uncertainty in Lx and Ly, because Lz has a non-zero expectation value. So it can't be that Lx squared and Ly squared have zero expectation value.

But let's be more precise about this. The expectation value of L squared is easy to calculate. It's h bar squared l, plus 1. Because we're in the state Ylm. This is the state Y l sub l, or ll. The expectation of Lz squared is equal to h bar squared, m squared. And m squared now, m is equal to l. So h bar squared, l squared. Aha. So the expected value of l square is not the same as Lx squared, but this is equal to the expected value of Lx squared plus Ly squared plus Lz squared.

Therefore the expected value of Lx squared plus the expected value of Ly squared is equal to the difference between this and this. We just subtract this off h bar squared ll plus 1 minus h bar squared, l squared, h bar squared l. And by symmetry you don't expect the symmetry to be broken between Lx and Ly. You can actually do the calculation and not just be glib about it. But both arguments give you the correct answer. The expectation value of Lx squared is equal to 1/2 h bar squared l. And ditto for y, in this state.

So notice two things. First off is we make the total angular momentum little l large. The amount by which we fail to have all the angular momentum in the z direction is getting larger and larger. We're increasing the crappiness of putting all the angular momentum in the z direction. However, as a ratio of the total angular momentum divided by L squared, so this divided by L squared, and this is h bar squared, l, l plus 1, and in particular this is l squared plus l, so if we took the ratio, the rational mismatch is getting smaller and smaller. And that's good. Because as we go to very

large angular momentums where things should start getting classical in some sense, we should get back the familiar intuition that you can put all the angular momentum in the z direction. Yeah?

AUDIENCE: Why are we imposing the [INAUDIBLE] condition that the expectation values of Lx and Ly should be identical?

PROFESSOR: Excellent. That's why I was saying this is a glib argument. You don't have to impose-- that needs to be done a little bit more delicately. But we can just directly compute this. And you do so on your problem set. Yeah?

AUDIENCE: Why doesn't the existence of the l equals 0 eigenfunction where the angular momentum the L squared is definitely 0 violate the uncertainty principle?

PROFESSOR: Awesome. On your problem set you're going to answer that, but let me give you a quick preview. This is such a great response. Just Let me give you a quick preview.

So from that Lx-- OK, this is a really fun question. Let me go into it in some detail. Wow, I'm going fast today. Am I going way too fast? No? A little? Little too fast? OK, ask me more questions to slow me down. I'm excited. I didn't get much sleep last night.

One of the great joys of being a physicist is working with other physicists. So yesterday one of my very good friends and a collaborator I really delight in talking with came to visit. And we had a late night dinner. And this led to me late at night, not doing my work, but reading papers about what our conversation was about. And only then at the very end, when I was just about to die did I write the response to the reviewer on the paper that I was supposed to be doing by last night's deadline. So I'm kind of tired, but I'm in a really good mood. It's a really good job.

Now I've totally lost my train of thought. What was the question again?

AUDIENCE: The existence of the l equals 0 state where the total angular momentum is definitely--

PROFESSOR: Excellent, and the uncertainty. So the question is, why don't we violate the

uncertainty when we know that L equals-- where am I; I just covered it-- when we know that L equals 0 and m equals 0, doesn't that destroy our uncertainty? Because we know that the angular momentum Lz is 0. Angular momentum for Lx is 0. Angular momentum for Ly is 0. All of them vanish. Doesn't that violate the uncertainty principle?

So let's remind ourselves what the form of that uncertainty relation is. The form of the uncertainty relation following from Lx Ly is i h bar Lz. Recall the general statement. The uncertainty in A times the uncertainty in B, squared, squared, is equal to-- let me just write it as h bar upon 2-- sorry, 1/2. The absolute value of the expectation value of the commutator, A with B. Good lord.

Dimensionally, does this work? Yes. OK, good. Because units of A, units of B. Unites of A, units of B. Triumph. And these are going to be quantum-mechanically small, because commutators have h bars. And commutators have h bars why? Not because God hates us. Why do commutators have h bars? What happens classically? In classical mechanics, do things commute? Yes. Why are the h bars and commutators physical observables? Because we exist. Because there's a classical limit.

OK so this is going to make quantum-mechanically small, so we expect the uncertainty relation to also be quantum-mechanically small. Just important intuition. So let's look at the specific example of Lx, Ly, and Lz. So the uncertainty in Lx, in any particular-- remember that this is defined as the uncertainty in a particular state psi, in a particular state psi. And this expectation value is taken in a particular state psi, that same state.

So the uncertainty of Lx, in some state Ylm, times the uncertainty of Ly in that same state Ylm should be greater than or equal to 1/2 the absolute value of the expectation value of the commutator of Lx and Ly. But the commutator of Lx and Ly is i h bar Lz. And i, when we pull it through this absolute value, is going to give me just 1. h bar is going to give me h bar. So h bar upon 2, expectation value of Lz, absolute value. Yeah?

Can Lx and Ly have zero uncertainty? When? Expectation value of Lz is 0. So that sounds good. It sounds like if Lz has expectation value of 0, then we can have Lx and Ly, definite. But that's bad. Really? Really, can we do that? Why not?

AUDIENCE: [INAUDIBLE]

[LAUGHTER]

PROFESSOR: Bless you, my son. Can we have Lx and Ly both take definite values, just because Lz? Why? What else do we have to satisfy? What other uncertainty relations must we satisfy? There are two more. And I invite you to go look at what those two more are and deduce that this is only possible if Lx squared, Ly squared, and Lz squared all have zero expectation value. In fact, I think it's just a great question that I think it's on your problem set.

So thank you for that question. It's a really good question. There was another question in here. Yeah?

AUDIENCE: Something that you said earlier [INAUDIBLE] half integer. So were you deriving this by counting the number of equations and somehow asserting-- Why does--?

PROFESSOR: Great. So the question is, wait, really? Why is L and integer a half integer. That was a little too quick. Is that roughly the right statement?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Good. Excellent. Let me go through the logic. So the logic goes like this. I know that the Ylm's-- if the Ylm's are eigenfunctions of L squared and Lz, and I've constructed this tower of them using the raising and lowering operators, we've already shown that the largest possible value of m is l and the least possible value is minus l. And these states must be separated by integer steps in m. OK, good.

So pick a value of l, a particular tower. And let the number of states in that tower be N sub l. So there's N sub l of them. Great. And what's the distance between these guys? We haven't assumed l is an integer yet. We haven't assumed that. So this N

sub l is an integer. Because it's the number of states. And the number of states can't be a pi.

Now let's count that-- that so how many states are there? There are Nl. But if I count one, two, three, four, the total angular momentum down here is 2l. I had some pithy way of giving this a fancy name. But I can't remember what it was. So the length of the tower in units of h bar, the height of this tower is 2l. But the number of steps I took in here was Nl minus 1. And that number of steps is times 1. So we get that 2l is N minus 1.

There's nothing fancy here. I'm just saying if L is 0 we go from here to here. There's just one element. So number states is 1, L is 0. So it's that same logic just repeated for every value of L. Other questions?

Coming back to this, something on this board should cause you some serious physical discomfort. We've talked about the l equals 0 m equals 0 state. This is a state which has no angular momentum whatsoever, in any direction at all. We've talked about the l equals 1 m equals 0 state. It carries no angular momentum in the z direction, but it presumably has non-zero expectation value for L squared x and a Ly squared. These guys are also fine. What about these guys?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yes! That's disconcerting. Do I have to have angular momentum in the z direction?

AUDIENCE: [INAUDIBLE]

[LAUGHTER]

PROFESSOR: That should go on a shirt somewhere. Let me ask the question more precisely, or in a way that's a little less threatening to me. Do you have to have angular momentum in the z direction? I'm sorry, what's your name? Does David need to have angular momentum in the z direction?

AUDIENCE: Yes.

PROFESSOR: Does this chalk need to have angular dimension in the-- well, the chalk's-- OK, classically no. It can have some total angular momentum, which is 0, and it can be rotating not in the z direction. It can be rotating in the zx plane. If it's rotating in the zx plane, it's got total angular momentum L squared as non-zero. But its angular momentum in the z direction is 0. Its axis is exactly along the y direction.

And so it's got no angular moment-- that's perfectly possible classically. And that's perfectly possible when L is an integer. Similarly when L is 2. This is the particular tower that I love the most. 2, 1, 0, minus 1, minus 2. The reason I love this the most is that it's related to gravity, which is pretty awesome. That's a whole other story. I really shouldn't have said that. That's only going to confuse you.

So for any integer, it's possible to have no angular momentum in the z direction. That means it's possible to rotate around the x-axis or the y-axis, orthogonal to the z-axis. That make sense? But for these half integer guys, you are inescapably spinning. There is no such thing as a state of this guy that carries no angular momentum. Anything well described by these quantum states is perpetually rotating or spinning. It carries angular momentum, we say precisely. Perpetually carries angular momentum in the z direction. Any time you measure it, it carries an angular momentum, either plus a half integer or minus a half integer. But never, ever zero.

AUDIENCE: But in the classical limit where you have very large L, the m equals one half state, or the m equals minus a half state is going to get arbitrarily small compared to the angular momentum. So isn't it just like where we said, well, OK, you can never have your angular momentum only in the z direction, but we don't care? Because in the classical limit it gets arbitrarily close to there.

PROFESSOR: See, one of the nice things about writing lectures like this is that you get to leave little landlines. So this is exactly one of those. Thank you for asking this question. Let me rephrase that question. Look, we all took high school chemistry. We all know about spin. The nuclei have spin. They have some angular momentum. But if you build up a lot of them, you build up a piece of chalk, look, as we said before, while it's true that there's some mismatch in the angular momentum in the z direction for

some large L state, it's not only angular momentum. Some is in Lz, Lx, and Ly as well. It's preposterously small for a macroscopic object where L is macroscopic. It's the angular momentum in Planck units, in units of the Planck constant. 10 to the 26th-- something huge.

Why would we even notice? But here's the real statement. The statement isn't just that this is true of macroscopic objects. But imagine you take a small particle. Imagine you take a single atom. We're deep in the quantum mechanical regime. We're not in the classical regime. We take that single atom. And if it carries angular momentum and it's described by L equals 1/2 state, that atom will never, ever, ever be measured to have its angular momentum in the z direction, or indeed any direction, be 0. You will never, in any direction, measure its angular momentum to be 0. That atom perpetually carries angular momentum. And that is weird.

OK, maybe you don't find it weird. This is good. You've grown up in an era when that's not a weird thing. But I find this deeply disconcerting. And you might say, look, we never actually measure an atom. But we do. We do all the time. Because we measure things like the spectre of light, as we'll study when we study atoms in a week, a couple weeks because of the exam-- sorry guys, there has to be an exam-- as we will find when we study atoms in more detail, or indeed, at all, we'll be sensitive to the angular momentum of the constituents of the atom. And we'll see different spectra. So it's an observable property when you shine light on gases. This is something we can really observe.

So what this suggests is one of two things. Either these are just crazy and ridiculous and we should ignore them, or there's something interesting and intrinsically quantum-mechanical about them that's not so familiar. And the answer is going to turn out to be the second, the latter of those. The ladder? That was not intentional. Maybe it was subconscious.

OK, so I want to think about some more consequences of the structure of the Ylm's and the eigenvalues, in particular of this tower structure. I want to understand some more. What other physics can we extract from this story? First, a very useful thing is

just to get a picture of these guys in your head. Let's draw the angular momentum eigenfunctions.

So what does that mean? Well first, when we talked about the eigenfunctions of momentum, linear momentum in one dimension, we immediately went to the wave function. We talked about how the amplitude to beat a particular spot varied in space. And the amplitude was just e to the iKX. So the amplitude was an oscillating-- the phase rotated. And the absolute value of the probability density was completely constant. Everybody cool with that? That was the 1D plane wave.

So the variables there, we had an angular momentum eigenstate. And that's a function of the position. Or, sorry, a linear momentum eigenstate. And that's a function of the position. Angular momentum eigenstates are going to be functions of angular position.

So I want to know what these wave functions look like, not just the eigenvalues. But I want to know what is the wave function associated to eigenvalues little l and little m look like, y sub lm of the angles theta and phi. What do these guys look like? How do we get them? This is going to be your goal for the next two minutes.

So the first thing to notice is that we know what the form of the eigenvalues and eigenfunctions are. If we act with Lz we get h bar m back. If we act with L squared we get h bar squared ll plus 1 back. But we also have other expressions for Lz and L squared. In particular-- I wrote them down last time in spherical coordinates. So I'm going to working in the spherical coordinates where the declination from the vertical is an angle theta, and the angle around the equator is an angle phi. And theta equals 0 is going to be up in the z direction. It's just a choice of coordinates. There's nothing deep. There's nothing even shallow. It's just definitions.

So we're going to work in the spherical coordinates. And in spherical coordinates we observe that this angular momentum, just following the definition from r cross p, takes a particularly simple form. That's a typo. h bar upon i, dd phi. And instead of writing L squared I'm going to write L plus minus, because it's shorter and also because it's going to turn out to be more useful. So this takes the form h bar, e to

the plus minus i phi, d theta, plus or minus cotangent of theta, d phi. So these are the expressions for Lz and L plus minus in spherical coordinates, in these spherical coordinates.

I want to know how Ylm depends on theta and phi. And it's clearly going to be easier to ask about the Lz eigenequation. So let's look at that. Lz on Ylm gives me h-- Oh yeah?

AUDIENCE: Is it h or h bar?

PROFESSOR: Oh, Jesus. When I write letters by hand, which is basically when I write to my mom, all my h's are crossed. I can't help it. So this is like the inverse. It's been a long time since I made that mistake. It's usually the other. So Lz acting on Ylm gives me h bar m, acting on Ylm. But that's what I get when I act with Lz, so let's just write out the differential equation .

So h bar m Ylm where m is an integer, or a half integer, depending on whether l is an integer or half integer. h bar m Ylm is equal to h bar upin i, d phi of Ylm. Using the awesome power of division and multiplication I will divide both sides by h bar. And I will multiply both sides by i. And we now have the equation for the eigenfunctions of Lz, which we actually worked on last time. And we can solve this very simply.

This says that, remember, Ylm is a function of theta and phi. Here we're only looking at the phi dependence, because that's all that showed up in this equation. So this tells us that the eigenfunctions Ylm are of the form of theta and phi, are of the form e to the im phi times some remaining dependence on theta, which I'll write as p of theta. And that p could depend on l and m. Cool?

Already, before we even ask about that dependence on theta, the p dependence, we learn something pretty awesome. Look at this wave function. Whatever else we know, its dependence on phi is e to the im phi. Now, remember what phi is. Phi is the angle around the equator. So it goes from 0 to 2 pi. And when it comes back to 2 pi it's the same point. Phi equals 0 and phi equals pi are two names for the same

point. Yes?

But that should worry you. Because note that as a consequence of this, Ylm of theta 0 is equal to, well, whatever it is. Sorry, theta of 2 pi is equal to e to the im 2 pi times Plm to theta. But this is equal to-- oh, now we're in trouble. If m is an integer, this is equal to e to the i integer 2 pi 1. Yes, OK. You're supposed to cheer at that point. It's like the coolest identity in the world.

So e to the i 2 pi, that's one. So if m is an integer then this is just Plm of theta, which is also what we get by putting in phi equals 0. So this is equal to Ylm of theta, comma, 0. But if m is a half integer then e to the i half integer times 2 pi is minus 1. And so that gives us minus Ylm of theta and 0, if m is a half integer.

So let me say that again, in the same words, actually. But let me just say it again with different emphasis. What this tells us is that Ylm at 0, as function of the coordinates theta and phi, Ylm-- so let's take m as an integer. Ylm at theta and 0 is equal to Ylm at theta and 0. That's good.

But if Y is a half integer, then Ylm at theta and 2 pi, which is the same point as Ylm at theta and 0, is equal to minus Ylm at theta and 0. That's less good. What must be true of Ylm, of theta and 0? 0. And was there anything special about the point 0? No. I could have just taken any point and rotated it around by pi. So this tells us that Ylm of theta and phi is identically equal to 0 if m is a half integer.

Huh. That's bad. Because what's the probability density of being found at any particular angular position? 0. Can you normalize that state? No. That is not a state that describes a particle. That is a state that describes the absence of a particle. That is not what we want. So these states cannot describe these values of l and m, which seem like perfectly reasonable values of l and m, perfectly reasonable eigenvalues of L squared and lm.

They cannot be used to label wave functions of physical states corresponding to wave functions on a sphere. You can't do it. Because if you try, you find that those wave functions identically vanish. OK? So these cannot be used. These do not

describe wave functions of a particle in quantum mechanics. They cannot be used. Those values, those towers cannot be used to describe particles moving in three dimensions. Questions about that? This is a slightly subtle argument. Yeah?

AUDIENCE: You said something earlier about how atoms would never have any zero angular momentum. And so the ones that have no zero angular momentum, we just said they're not possible. So [INAUDIBLE]?

PROFESSOR: Excellent question. I said it slightly diff-- so the question is, look, earlier you were saying, yeah, yeah, yeah. There are atoms in the world and they have half integer angular momentum. And you can shine a light on them and you can tell and stuff. But you just said these can't exist. So how can those two things both be true?

Thank you for this question. It's a very good question. I actually said a slightly different thing. What I said was, states where angular momentum lm are half integers cannot be described by a wave function of the coordinates. We're going to need some different description. And in particular, we're going to need a different description that does what? Well, as we take phi from 0 to 2 pi, as we rotate the system, we're going to pick up a minus sign.

So in order to describe an object with lm being a half integer, we can't use the wave function. We need something that is allowed to be doubly valued. And in particular, we need something that behaves nicely. When you rotate around by 2 pi, we need to come back to a minus sign, not itself. So at some object that's not a function, it's called a spinner. So we'll talk about it later. We need some object that does that.

So there's this classic demonstration at this point, which is supposed to be done in a quantum mechanics class. So at this point the lecturist is obliged to do the following thing. You say, blah, blah, blah, if things rotate by 2 pi they have to come back to themselves. And then you do this. I'm going to rotate my hand like a record by 2 pi. And it will not come back to itself. OK?

And it quite uncomfortably has not come back to itself. But I can do a further rotation to show you that it's a minus sign. I can do a further the rotation by 2 pi and have it

come back to itself. And I kept the axis vertical, yeah? OK, so at this point you're all supposed to go like, oh, yes, uh-huh, mmm. So now that we've got that out of the way, I have an arm.

So the story is a little more complicated than that. This is actually a fair demonstration, but it's a slightly subtle story. If you want to understand it, ask your recitation instructor or come to my office hours.

AUDIENCE: What about USB sticks?

PROFESSOR: USB sticks?

AUDIENCE: You insert them here and they don't go and insert the other way.

[LAUGHTER]

PROFESSOR: What about USB sticks? You insert them this way, they don't work. You insert them this way, they don't work. But if you do it again, then they do.

[LAUGHTER]

[APPLAUSE]

PROFESSOR: OK. That's pretty good. So for the moment, as long as we want to describe our system with a wave function of position, which means we're thinking about where will we find it as a function of angle, we cannot use the half integer l or m. So if we can't use the half interger l or m, fine. We'll just throw them out for the moment and we'll use the integer l and m. And let's keep going.

What we need to determine now is the P. We've determined the phi dependence, but we need the theta dependence. We can get the theta dependence in a sneaky fashion. Remember the harmonic oscillator in 1D. When we wanted to find the ground state wave function, we could either solve the energy eigenvalue equation, which is a second order differential equation and kind of horrible, or we could solve

the ground state equation that said that the ground state is annihilated by the annihilation operator, which is a first order difference equation and much easier to solve. Yeah?

Let's do the same thing. We have an annihilation condition. If we have the top state, L plus on Y ll is equal to 0. But L plus is a first order differential operator. This is going to be easier. So we need to find a solution to this equation. And do I want to go through this? Yeah, why not. OK, so I want this to be equal to 0. But L plus Yll is equal to-- well, it's h bar, e to the plus i phi, d theta, plus cotangent theta, d phi on Yll. And Yll is e to the i l phi times Plm of theta. Cool?

So dd phi on e to the il phi, there's no phi dependence here. It's just going to give us a vector of il. Did I get the-- yeah. So that's going to give us a plus il and no dd phi. And this e to the il phi we can pull out. But this has to be equal to 0. So this says that 0 is equal to d theta plus il cotangent theta P lm of theta.

And this is actually much better than it seems for the following reason. dd theta-- find the dd theta. Cotangent of theta is cosine over sine. That's what you get if you take the derivative of lots of sine functions-- sine to the l, say. Take a derivative. You lose a power of sine and you pick up a power of cosine. So multiplying by cotangent gets rid of a power of sign and gives you a power of cosine, which is a derivative of sine.

So Plm, noticing the l-- and I screwed up an i somewhere. I think I wanted an i up here. Let's see. i. Sorry. Yes, I want an i cotangent. OK, that's much better. So i cotangent-- good, good. And then the i squareds give me a minus l. And this tells us that Plm, so if this is sine to the l, then d theta gives us an l sine to the l minus 1 cosine, which is what I get by taking sine to the l and multiplying by cosine, dividing by sine, and multplying by l.

So this gives me Pll. This is for the particular state ll. We're looking at the top state and we're annhilating it. So Pll is equal to some coefficient, so I'll just say proportional to sine to the l of theta. So this tells us that Yll is equal to some normalization, sub ll, just some number, times from the phi dependence e to the il

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phi, and from the theta dependence, sine to the l of theta.

So this is the form. Sorry, go ahead.

AUDIENCE: What's the symbol there?

PROFESSOR: Oh, this twisted horrible thing? It's proportional to. But it was a long night. So now we have the wave function explicitly as a function phi and as a function of theta completely understood for the top state in any tower. This is for any L. The top state in any tower is e to the il phi. Does that make sense? Well Lz is h bar upon id phi. So that gives us h bar as m as l. So that's good. That's the top state. And from the sine theta we just checked. We constructed that this indeed has the-- well, if you then check you will find that the L squared eigenvalue, which you'll do on the problem set, the l squared eigenvalue is h bar squared ll plus 1.

AUDIENCE: Quick question. The expression we have for the L plus minus operators, how did we construct the expressions for Lx and Ly?

PROFESSOR: Good. It's much easier than you think. So Lx is equal to-- L is r cross b, right? So this is going to be yPz minus zPy. And this is equal to h bar upon i, ydz minus zdy. But you know what y is in spherical coordinates. And you know what derivitive with respect to z is in spherical coordinates. Because you know what z is, and you know the chain rule. So taking this and just plugging in explicit expression for the change of variables to spherical coordinates takes care of it. Yeah?

AUDIENCE: What does the superscript of the sine indicate? Is that sine to the power of l?

PROFESSOR: Sorry. This is bad notation. It's not bad notation, it's just not familiar notation. It's notation that is used throughout theoretical physics. It means this, sine theta to the l. The lth power of sine. For typesetting reasons we often put the power before the argument. Yeah, no, it's a very good question. Thank you for asking, because it was unclear. I appreciate that. Other questions?

AUDIENCE: [INAUDIBLE] L hat [INAUDIBLE]?

PROFESSOR: How did we come up with the L hat plus minus? That was from this. So we know

what the components of the angular momentum are in Cartesian coordinates. And you know how, because it's coordinates, to change variables from Cartesian to spherical. So you just plug this in for Lx. But L plus is Lx plus ioy, and so you just take these guys in spherical form and add them together with the relative i. And that gives you that expression.

AUDIENCE: Maybe I missed this, but can you just explain the distinction between Y sub ll and Y sub lm?

PROFESSOR: Yeah, absolutely. So Y sub ll, it means Y sub lm where m is equal to l.

AUDIENCE: Oh, OK. It was just the generic [INAUDIBLE].

PROFESSOR: It's a generic eigenfunction of the angular momentum, with the angular momentum in the z direction being equal to the angular momentum in the total angular momentum. At least for these numbers. OK? Cool.

Good, so now, if we know this, how do we just as a side note-- suppose we know-- well, suppose we know this? We know this. We know what the top state in the tower looks like. How do I get the next state down in the tower? how do I get Yl l minus 1?

AUDIENCE: Lower it.

PROFESSOR: Lower it. Exactly. So this is easy, L minus on Yll. And we have to be careful about normalization. So again, it's proportional to. But this is easy. We don't have to solve any difference equations. We just have to take derivatives. So it's just like the raising operator for a harmonic oscillator. We can raise and lower along the tower and get the right wave function.

To give you some examples-- yeah, let's do that here. Let me just quickly give you a few examples of the first few spherical harmonics. Sorry, I should give these guys a name. These functions, Ylm of theta and phi, they're called the spherical harmonics. They're called these because they solve the Laplacian equation on the sphere, which is just the eigenvalue equation. L squared on them is equal to a constant times those things back.

Just to tabulate a couple of examples for you concretely, consider the l equals 0 states. What are the allowed values of m for little l equals 0? 0. So Y0,0 is the only state. And if you properly normalize it, it's 1 over root 4 pi. OK, good. what about l equals 1? Then we have Y1, 0 and we have Y1 minus 1. And we have Y1,1. So these guys take a particularly simple form. Root 3-- I'm not even going to worry about the coefficient. They're in the notes. You can look them up anywhere.

So first off let's look at Y1, 1. So non-linear today. So Y1, 1, it's going to be some normalization. And what is the form? It's just e to the il phi sine theta to the l. l is 1. So this is some constant times e to the i phi, sine theta. Who 1, 0? Well, it's got angular momentum 0 in the z direction, in Lz. So that means how does it depend on phi? It doesn't. And you can easily see that, because when we lower we get an e to the i minus i phi.

Anyway, so this gives us a constant times no e to the i phi, no phi dependents, and cosine of theta. And if you get a cosine of theta, the d theta and the cotangent d phi will give you the same thing. And Y1 minus 1 is again a constant, times e to the minus i phi. So it's got m equals minus 1 and sine theta again. Notice a pleasing parsimony here. The theta dependence is the same for plus m and minus m.

So what about Y2, 2? Some constant e to the i 2 phi, sine squared theta, dot, dot, dot, Y0, 0. And here it's interesting. Here we just got one term from taking the derivative. They both give you cosine. But now there are two ways to act with the two derivatives. And this gives you a constant. Now what's the phi dependence? It's nothing, because it's got m equals 0. And so the only dependence is on theta. And we get a cos squared whoops, there's a 3-- 3 cos squared theta minus 1.

AUDIENCE: Do you mean Y2, 0?

PROFESSOR: Oh, shoot. Thank you. Yes, I mean Y2, 0. Thank you. And then if we continue lowering to Y2 minus 2, this is equal to, again, a constant. And the it's going to be the same dependence on theta, but a different dependence of phi. e to the minus 2i phi, sine squared theta. OK? Yeah.

AUDIENCE: Aren't these not normalizable?

PROFESSOR: Why?

AUDIENCE: Oh, never mind.

PROFESSOR: Good. So let me turn that into a question. The question is, are these normalizable? Yeah, so how would we normalize them? What's the check for if they're normalizable or not?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, we integrate them norm squared over a sphere. Not over a volume, just over a sphere. Because they're only wave functions on the sphere. We haven't dealt with the radial function. We'll deal with that later. That will come in the next lecture. Other questions? Yeah?

AUDIENCE: Can you explain one more time why m equals 0 doesn't have [INAUDIBLE]?

PROFESSOR: Yeah, why m equals 0 doesn't have any phi dependence? If m equals 0 had phi dependence, then we know that the eigenvalue of Llz is what we get when we take a derivative with respect to phi. But if the Lz eigenvalue is 0, that means that when we act with dd phi we get 0. That means it can't depend on phi. Cool? Other questions?

What I'm going to do next is I'm going show you, walk you through some of these angular momentum eigenstates, graphically on the computer. Before we do that, any questions about the calculation so far? OK.

So this mathematical package I'll post on the web page. And at the moment I think it's doing the real part. So what we're looking at now is the real part. Actually, let's look at the absolute value. Good. So here we are, looking at the absolute value of-- that's not what I wanted to do. So what we're looking at in this notation is some horrible parametric plot. You don't really need to see the mathemat-- oh, shoot. Sorry.

You don't need to see the code, particularly. So I'm not going to worry about it, but it will be posted. So here we're looking at the absolute value, the norm squared of the spherical harmonic Y. And the lower eigenvalue here, lower coordinate is the l. And the upper is m. So when l is 0, what we get-- here's what this plot is indicating. The distance away from the origin in a particular angular direction is the absolute value of the wave function. So the further away from the origin the colored point you see is, the larger the absolute value. And the color here is just to indicate depth and position. It's not terribly meaningful.

So here we see that we get a sphere. So the probability density or the norm squared of the wave function of the spherical harmonic is constant. So that makes sense. It's spherically symmetric. It has no angular momentum. As we start increasing the angular momentum, let's take the angular momentum l is 1, m is 0 state, now something interesting happens.

The total angular momentum is 1. And we see that there are two spheres. Let me sort of rotate this. So there are two spheres, and there's the z-axis passing through them. And so the probability is much larger around this lobe on the top or the lobe on the bottom. And it's 0 on the plane. Now, that's kind of non-intuitive if you think, well, Lz is large, so why is it along the vertical?

Why is that true? So here, this is the Lz equals 0 state. That means it carries no angular momentum along the z-axis. That means it's not rotating far out in the xy plane. So your probability of finding it in the xy plane is very small. Because if it was rotating in the xy plane it would carry a large angular momentum in Lz. But Lz is 0. So it can't be extended out in the xy plane. Cool?

On the other hand, it can carry angular momentum in the x or the y direction. But if it carries angular momentum in the x direction for example, that means the system is rotating around the z-- sorry. If it carries angular momentum in the x direction, it's rotating in the zy plane. So there's some probability to find it out of the plane in y and z. But it can't be in the xy plane. Hence it's got to be in the lobes up above. That cool?

So it's very useful to develop an intuition for this stuff if you're going to do chemistry or crystallography or any condensed manner of physics. It's just very useful. So I encourage you to play with these little applets. And I'll post this mathematics package.

But let's looked at what happens now if we crank up the angular momentum. So as we crank up the l angular momentum, now we're getting this sort of lobe-y thing, which looks like some sort of '50s sci-fi apparatus. So what's going on there?

This is the 2, 0 state. And the 2, 0 state has a 3 cos squared theta minus 1. But cos squared theta, that means it's got two periods as it goes from vertical to negative. And if you take that and you square it, you get exactly this. OK? So they're using the cos squared as a function of the angle of declination from vertical. And it's m equals 0, so you're saying no dependence on phi. Of course, that's a little cheap. Because the angular dependence on phi is just an overall phase. So we're not going to see it in the absolute value. Everyone agree on that? We're not going to see the absolute-- OK.

So let's check that. Let's take l2 and m2. So now when l is 2 and m is 2, we just get this donut. So what that's saying is, we've got some angular momentum. l is 2. But all the angular momentum, almost all of it, anyway, is in the z direction. And is that what we're seeing here? Well, yeah. It seems like it's most likely to find the particle, the probability is greatest, out in this donut around the plane.

Now, if it were Lz is equal to l, it would be flat. It would be a strictly 0 thickness pancake. But we have some uncertainty in what Lx and Ly are, which is why we got this fattened donut. Everyone cool with that? And if we crank up l and we make-- yeah, right? So you can see that you've got some complicated shapes. But as we crank up l and crank up m, we just get a thinner and thinner donut. And the fact that the donut's getting thinner and thinner is that l over L squared that we did earlier. Cool? It's still a donut, but it's getting relatively thinner and thinner, by virtue of getting wider and wider.

And a last thing to show you is let's take a look now at-- in fact, let's go to the l

equals 0 state. Let's take a look at the real part. So now we're looking at the real part. And nothing much changed for the Y0, 0. But for Y2, 0, well, still not much changed. For Y2, 0 let's now-- this is sort of disheartening. Nothing really has changed. Why? Because it's real, exactly. So for Y2, 0, as long as m is equal to 0, this is real. There's no phase. The phase information contains information about the Lz eigenvalue.

So we can correct this by changing the angular momentum. Let's-- oh shoot, how do I do that? I can't turn it off at the moment. OK, whatever. So here we have a large M. And now we've got this very funny lobe-y structure. So this is the Y2, 2, which a minute ago looked rotationally symmetric. And now it's not rotationally symmetric. It's this lobe-y structure, where the lobes are-- remember, previously we had a donut. Now we have these lobes when we look at the real part. How does that make sense?

Well, we've got an e to the i 2 phi. And if we look at the real part, that's cosine of 2 pi. And so we're getting a cosine function modulating the donut. And if we look at the real part, let's do the same thing. Let's look at the imaginary part. And the imaginary part of Y2, 0, we get nothing. That's good, because Y2, 0 was real. That would have been bad.

But if we look at the imaginary part of Y2, 2, we get the corresponding lobes, the other lobes, so that cos squared plus sine squared is 1. Play with these. Develop some intuition. They're going to be very useful for us when we talk about hydrogen and the structure of solids. And I will see you next week.

[APPLAUSE]