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PROFESSOR: So today we're going to do our last lecture on scattering in 1D quantum mechanics, and we're going to introduce some powerful ideas, in particular, the phase shift and the S matrix, and they're cool. We'll use them for good. Before we get started, questions from last time?

AUDIENCE: What was the music that was playing?

PROFESSOR: Just now? It's a band called Saint Germain. It's actually a guy, but he refers to himself as a band called Saint Germain. Anyway, it's from an album I think called "Traveler." Physics questions? Anyone? OK, good. Well, bad, actually. I'd much prefer it if you had questions, but I'll take that as a sign of knowledge, competence, and mastery.

So last time, we talked about scattering past a barrier with some height L and some height V , which I think we called V_0 , and we observed a bunch of nice things about this. First off, we computed the probability of transmission across this barrier as a function of the energy, and it had a bunch of nice features. First off, it asymptoted to 1 at high energies, which makes sense. Things shouldn't really care if you have a little tiny barrier.

It went to 0 at 0 energy. I think we should all find that obvious. If you have extremely low energy, you're just going to bounce off this very hard wall. In between, though, there's some structure. In particular, at certain values of the energy corresponding to certain values of the wave number in the barrier region, at certain values of the energy, we saw that the transmission was perfect. Oops. This should have been here. Sorry. The transition was perfect at special values of the energy, and that the reflection at those points was 0. You transmitted perfectly.

But meanwhile, the reflection, which is 1 minus the transmission, hit maxima at special points, and those special points turn out to be half integer shifted away from the special points for perfect transmission. So at certain points, we have perfect transmission. At certain points, we have extremely efficient reflection. One of our goals today is going to be to understand this physics, the physics of resonance in scattering off of a potential.

So the asymptotes we understand. Classically, that makes sense. Classically, that makes sense. But classically, this is a really weird structure to see. And notice that it's also not something that we saw when we looked at scattering off of a simple step. When we looked at a simple step, what we got was something that looked like 0 and then this. There was no structure. It was just a nice, simple curve.

So something is happening when we have a barrier as opposed to a step. So our job is going to be, in some sense, to answer why are they so different. Before we get going, questions about the step barrier. Yeah?

AUDIENCE: That little line that you drew over your other graph, is that to scale?

PROFESSOR: Oh, sorry. This one?

AUDIENCE: No, the little line that you drew right there.

PROFESSOR: This? Sorry, I should draw it separately. That was the transmission as a function of energy across a single step, and so that looked like this.

AUDIENCE: So if we were to overlap that onto our resonance one--

PROFESSOR: Exactly.

AUDIENCE: Would it [INAUDIBLE] to?

PROFESSOR: Oh, sorry. So I just meant to compare the two. I just wanted to think of them as two different systems. In this system, the transmission curve is a nice, simple curve. It has no structure. In the case of a step barrier, we get non-zero transmission at low energies instead of going to 0 and we get this resonant structure. So it's just for

contrast.

Before going into that in detail, I want to do one slight variation of this problem. I'm not going to do any of the computations. I'm just going to tell you how to get the answer from the answers we've already computed. So consider a square well. You guys have solved the finite square well on your problem sets. Consider a finite square well again of width L and of depth now minus V_0 . It's the same thing with V goes to minus V . And again, I want to consider scattering states. So I want to consider states with positive energy, energy above the asymptotic potential.

Now, at this point, by now we all know how to solve this. We write plain waves here, plain waves here, and plain waves in between. We solve the potential in each region where it's constant, and then we impose continuity of the wave function and continuity of the derivative of the wave function at the matching points. So we know how to solve this problem and deduce the transmission reflection coefficients. We've done it for this problem, and it's exactly the same algebra. And in fact, it's so exactly the same algebra that we can just take the results from this one and take V to minus V and we'll get the right answer. You kind of have to.

So if we do, what we get for the transition probability-- and now this is the transmission probability for a square well-- and again, I'm going to use the same dimensionless constants. g_0 squared is equal to $2mL$ squared over \hbar squared times V_0 . This is 1 over an energy. This is an energy. This is dimensionless. And the dimensionless energy ϵ is equal to E over V . To express my transmission probability, life is better when it's dimensionless.

So T , the transmission probability, again, it's one of these horrible 1 overs. 1 over 1 plus 1 over 4ϵ , ϵ plus 1 , sine squared of g_0 , square root of ϵ plus 1 . This is again the same. You can see what we got last time by now taking V to minus V again. That takes ϵ to minus ϵ . So we get ϵ , 1 minus ϵ , or ϵ minus 1 picking up the minus sign from this ϵ , and here we get a 1 minus ϵ instead of 1 plus ϵ . That's precisely what we got last time. So if you're feeling punchy at home tonight, check this. In some sense, you're

going to re-derive it on the problem set.

So it looks basically the same as before. We have a sine function downstairs, which again will sometimes be 0. The sine will occasionally be 0 when its argument is a multiple of π . And when that sine function is 0, then the transmission probability is $\frac{1}{1 + 0}$. It's also known as 1. Transmission is perfect. So we again get a resonant structure, and it's in fact exactly the same plot, or almost exactly the same plot.

I'm going to plot transmission-- let's go ahead and do this-- transmission probability as a function, again, of the dimensionless energy. And what we get is, again, not very well drawn resonances where transmission goes to 1 thanks to my beautiful artistic skills. Points where the reflection hits a local maximum.

But we know one more thing about this system, which is that, in addition to having scattering states whose transition probability are indicated by this plot, we know we also have, for negative energy, bound states. Unlike the step barrier, for the step well, we also have bound states. So here's the transmission curve, but I just want to remind you that we have energies. At special values of energies, we also have bound states.

And precisely what energies depends on the structure of the well and the depth, but if the depth of the well is, say, V_0 so this is minus 1, we know that the lowest bound state is always greater energy than the bottom of the well. Cool? So epsilon, remember, is E over V_0 , and in the square well of depth V_0 , the lowest bound state cannot possibly have energy lower than the bottom of the well, so its epsilon must have epsilon greater than minus 1.

Anyway, this is just to remind you that there are bound states. Think of this like a gun put down on a table in a play in Act One. It's that dramatic. It will show up again. It will come back and play well with us.

I want to talk about these resonances. Let's understand why they're there. There are a bunch of ways of understanding why these resonances are present, and let

me just give you a couple. First, a heuristic picture, and then I want to give you a very precise computational picture of why these resonances are happening. So the first is imagine we have a state where the transmission is perfect. What that tells you is KL is a multiple of π .

If this is the distance L of the well, in here the wave is exactly one period if its KL is equal to a multiple of π . Let me simplify my life and consider the case KL is 2π . Sorry. 4π . I switched notations for you. I was using in my head, rather in my notes, width of the well is $2L$ for reasons you'll see on the notes if you look at the notes, but let's ignore that. Let's focus on these guys.

So consider the state, configuration in energy, such that in the well, we have exactly one period of the wave function. That means that the value of the wave function at the two ends is the same and the slope is the same. So whatever the energy is out here, if it matches smoothly and continuously and its derivative is continuous here, it will match smoothly and continuously out here as well with the same amplitude and the same period inside and outside.

The amplitude is the same and the phase is the same. That means this wave must have the same amplitude and the same period. It has to have the same period because it's the same energy, but it must have the same amplitude and the same slope at that point. And that means that this wave has the same amplitude as this wave. The norm squared is the transmission probability, the norm squared of this divided by the norm squared of this. That means the transmission probability has to be 1.

What would have happened had the system, instead of being perfectly periodic inside the well-- actually, let me leave that up. Let's do it as this. Shoot. I'm even getting my qualitative wave functions wrong. Let's try this again. Start at the top, go down, and then it's deeper inside the well so the amplitude out here, the difference in energy between the energy and the potential, is less, which means the period is longer and the amplitude is greater.

The way I've drawn it, it's got a particular value. It's got zero derivative at this point,

so it's got to-- so there's our wave function. Same thing out here. The important thing is the amplitude here has to be the same as the amplitude here because the amplitude and the amplitude and the phase were exactly as if there had been no intervening region. Everyone agree with that?

By contrast, if we had looked at a situation where inside the well, it was not the same amplitude, so for example, something like this, came up with some slope which is different and a value which is different, then this is going to match onto something with the same period but with a different amplitude than it would have over here. Same period because it's got the same energy, but a different amplitude because it has to match on with the value and the derivative. So you can think through this and pretty quickly convince yourself of the necessity of the transmission aptitude being 1 if this is exactly periodic.

Again, if this is exactly one period inside, you can just imagine this is gone and you get a continuous wave function, so the amplitude and the derivative must be the same on both sides as if there was no barrier there. When the wave function doesn't have exactly one period inside the well, you can't do that, so the amplitudes can't be the same on both sides. But that's not a very satisfying explanation. That's really an explanation about solutions to the differential equation. I'm just telling you properties of second order differential equations.

Let's think of a more physical, more quantum mechanical explanation. Why are we getting these resonances? Well, I want to think about this in the same way as we thought about the boxes in the very first lecture.

Suppose we have this square well, and I know I have some amplitude here. I've got a wave function. It's got some amplitude here. It's got some momentum going this way, some positive momentum. And I want to ask, what's the probability that I will scatter past the potential, let's say to this point right here just on the other side of the potential? What's the probability that I will scatter across?

Well, you say, we've done this calculation. We know the probability to transmit across this step potential. We did that last time. So that's T step. We know the

probability of scattering across this potential step, and that's transmission up. And so the probability that you transmit from here to here is the probability that you transmit here first and then the probability that you transmit here, the product of the probabilities. Sound reasonable?

Let's vote. How many people think that this is equal to the transmission probability across the potential well? Any votes? You have to vote one way or the other. No, it is not. How many people vote? OK. Yes, it is. OK. The no's have it, and that's not terribly surprising because, of course, these have no resonance structure, so where did that come from if it's just that thing squared? So that probably can't be it, but here's the bigger problem. Why is this the wrong argument?

AUDIENCE: Because there's reflection.

PROFESSOR: Yeah, exactly. There's reflection, but that's only one step in the answer or why it's the wrong answer. Why else?

AUDIENCE: Your transmission operations have to do it far away?

PROFESSOR: That's, true but I just want the probability, and if I got here with positive momentum, I'm eventually going to get out to infinity, so it's the same probability because it's just an E to the ikx out here. The wave function is just E to the ikx so the probability is going to be the same. Other reasons?

AUDIENCE: Well, the width of the well is very important, but the first argument ignores that.

PROFESSOR: Exactly. That's also true. So far, the reasons we have are we need the width of the well, doesn't appear. That seems probably wrong. The second is it's possible that you could have reflected. We haven't really incorporated that in any sort of elegant way.

AUDIENCE: There are other ways to transmit by reflecting twice.

PROFESSOR: That's absolutely true. There are other ways to transmit, so you could transmit, then you could reflect. So we could transmit then reflect, and reflect again, and then transmit. We could transmit, reflect, reflect, transmit. What else? Do probabilities

add in quantum mechanics? And when you have products of events, do probabilities multiply? What adds in quantum mechanics?

AUDIENCE: [INAUDIBLE].

PROFESSOR: The amplitude. The wave function. We do not take the product of probabilities. What we do is we ask, what's the amplitude to get here from there, and we take the amplitude norm squared to get the probability. So the correct question is what's the amplitude to get from here to here? How does the wave function of the amplitude change as you move from here to here?

And for that, think back to the two slit experiment or think back to the boxes. We asked the following question. The amplitude that you should transmit across this well has a bunch of components, is a sum of a bunch of terms. You could transmit down this well. Inside here, you know your wave function is $e^{i k' L}$ or I think I'm calling this $k_2 x$.

And in moving across the well, your wave function evolves by an $e^{i k_2 L}$, and then you could transmit again with some transmission amplitude. So this would be the transmission down times $e^{i k_2 L}$ times the transmit up. As we saw last time, these are the same, but I just want to keep them separate so you know which one talking about.

This is a contribution. This is something that could contribute to the amplitude. Is it the only thing that could contribute to the amplitude? No. What else could contribute? Bounce, right?

So to get from here, to here I could transmit, evolve, transmit. I could also transmit, evolve, reflect, evolve, reflect, evolve, transmit. So there's also a term that's $t, e^{i k L}$, $r, e^{-i k L}$, $r, e^{i k L}$. Sorry, $e^{i k L}$ because we're increasing the evolution of the phase. And then transmit finally at the end. And these k 's are all k_2 , but I could have done that many times.

But notice that each time what I'm going to do is I'm going to transmit, reflect,

reflect, transmit or transmit, reflect, reflect, reflect, reflect, transmit. So I'm always going to do this some number of times. I do this once, I do this twice, I do it thrice. This gives me a geometric series. This is $t e^{-i k^2 L}$ times $1 +$ this quantity plus this quantity squared. That's a geometric series, 1 over $1 +$ this quantity squared. Sorry, $1 -$ because it's a geometric sum.

And what is this quantity? Well, it's r squared, and remember the r 's in both directions are the same, so I'm just going to write it as r squared $e^{-2i k^2 L}$. It's a real number, but I'm going to put the absolute value on anyway. It's going to simplify my life. $e^{-2i k^2 L}$. So this is our prediction from multiple bounces.

What we're doing here is we're taking seriously the superposition principle that says given any process, any way that that process could happen, you sum up the amplitudes and the probability is the norm squared. If we have a source and we have two slits and I ask you, what's the probability that you land here, the probability is not the sum of the probabilities for each individual transit. The probability is the square of the amplitude where the amplitude is the sum, amplitude top plus amplitude bottom. Here, there are many, many slits. There are many different ways this could happen. You could reflect multiple times. Everyone cool with that?

By the same token, we could have done the same thing for reflection, but let's stick with transmission for the moment. This is what we get for the transmission, and again, the transmission amplitudes across the step, as we saw last time, are the same. So this is, in fact, t squared. And this gives us a result for transmission down the potential well, and if we use what the reflection and transmission amplitudes were for our step wells, the answer that this gives is 1 over $e^{-i k^2 L} - 2i$ upon the transmission for a step times sine of $k^2 L$.

Now, this isn't the same as the probability that we derived over here, but that's because this isn't the probability. This was the amplitude. We just computed the total amplitude. To get the probability, we have to take the norm squared of the amplitude. And when we take the norm squared, what we get is 1 upon $1 +$

over 4ϵ , $\epsilon + 1$, \sin^2 of $g_0 \sqrt{\epsilon + 1}$. We get the same answer.

AUDIENCE: Is that an equality?

PROFESSOR: This is an equality. Oh, sorry. Thank you, Barton. We get to the transmission probability when we take the norm squared of the amplitude. Thank you. Is equal to this, which is the same as we got before. Thanks. Yeah, please?

AUDIENCE: Tell me why kL equals π doesn't work.

PROFESSOR: kL equals π doesn't work. It does. You just have to be careful what kL --

AUDIENCE: On the first drawing, you changed your kL [INAUDIBLE].

PROFESSOR: Because I'm an idiot. Because I got a factor of 2 wrong. Thank you. Thank you, Matt. Thank you. Answer analysis. It's a wonderful thing.

This does something really nice for us. Why are we getting a resonance at special values at the energy? What's happening? Well, in this quantum mechanical process of multiple interactions, multiple scatterings, there are many terms in the amplitude for transmitting.

There are terms that involve no reflection, there are terms that involve two reflections, there are terms that involve four reflections, and they all come with an actual magnitude and a phase. And when the phase is the same, they add constructively, and when the phases are not the same, they interfere. And when the phases are exactly off, they interfere destructively, and that is why you're getting a resonance. Multiple terms in your superposition interfere with each other, something that does not happen classically. Classically, the probabilities are products. Quantum mechanically, we have superposition and probabilities are the squares of the amplitude, and we get interference effects in the probabilities. Cool?

To me, this is a nice, glorious version of the two slit experiment, and we're going to bump up again into it later when we talk about the physics of solids in the real world. Questions at this point? Yeah?

AUDIENCE: Question on something you said earlier. What is the $k_2 L$ equals [INAUDIBLE]?

PROFESSOR: Yeah, what's special about that? What's special about that is at this point, where kL is $n\pi$, we get perfect transmission. When kL is equal to $n + 1/2\pi$, the reflection is as good as it gets. What that's really doing is it's saying, when is this locally largest? So that's the special points when the transmission is as small as possible, which means the reflection, which is 1 minus the transmission probability, is as large as possible. And you can get that, again, from these expressions. Other questions?

So I want to think a little more about these square barriers. And in particular, in thinking about the square well barrier, what we've been talking about all along are monochromatic wave packets. We've been talking about plain waves, just simply e^{ikx} , but you can't put a single particle in a state which is a plain wave. It's not normalizable. What we really mean at the end of the day when we talk about single particles is we put them in some well localized wave packet, which at time 0 , let's say, is at position x_0 , which in this case is negative, and which has some well defined average momentum, k_0 . I'll say it's the expectation value of momentum in this wave packet.

And the question we really want to ask when we talk about scattering is, what happens to this beast as it hits the barrier, which I'm going to put the left hand side at 0 and the right hand side at L and let the depth be minus v_0 again. What happens as this incident wave packet hits the barrier and then scatters off? Well, we know what would happen if it was a plain wave, but a plain wave wouldn't be localized. So this is the question I want to ask, and I want to use the results that we already have.

Now, here's the key thing. Consider to begin with just a wave packet for a free particle centered at x_0 and with momentum k_0 , just look for a free particle. We know how to write this. We can put the system, for example, we can take our wave function at time 0 to be a Gaussian, some normalization times $e^{-\frac{x - x_0}{2a}}$. And we want to give it some momentum k_0 , and you know how to do that after the last problem set, $e^{ik_0 x}$. Everyone

cool with that?

So there's our initial wave function, and we want to know how it evolves with time, and we know how to do that. To evolve it in time, we first expand it in energy eigenstates. So $\psi(x, 0)$ is equal to-- well, the energy eigenstates in this case are plain waves, $\frac{1}{\sqrt{2\pi}} e^{ikx}$ times some coefficients, $f(k)$, the expansion coefficients. But these are just the Fourier transform of our initial Gaussian wave packet, and we know what the form of $f(k)$ is equal to.

Well, it's a Gaussian of width $1/\alpha$ upon $e^{-i(k-k_0)x}$ because it has momentum k_0 , so it's centered around k_0 , squared over $2\alpha^2$, and the α goes upstairs. And the position of the initial wave packet is encoded in the Fourier transform, and I'm going to put a normalization here, which I'm not going to worry about, with an overall phase e^{-ik_0x} .

So in just the same way that adding on a phase, e^{ik_0x} , in the position space wave function tells you what the expectation value of momentum is, tacking on the phase-- and we can get this from just Fourier transform-- tacking on the phase e^{ik_0x} in the Fourier transform tells you the center point, the central position, of the wave packet. So this you did on the problem set, and this is a trivial momentum space version of the same thing. So here's our wave packet expanded in plain waves, which are energy eigenstates. And the statement that these are energy eigenstates is equivalent to the statement that under time evolution, they do nothing but rotate by a phase.

So if we want to know what the wave function is as a function of time, $\psi(x, t)$ is equal to the integral, $\int dk$, the Fourier mode $f(k)$. I'll write it out. Actually, I'm going to write this out explicitly. $f(k) \frac{1}{\sqrt{2\pi}} e^{ikx - i\omega t}$ where ω , of course, is a function of k . It's a free particle, so for our free particle, $\hbar^2 k^2 / 2m$ is equal to $\hbar \omega$. We've done this before.

But now what I want to do is I want to take exactly the same system and I want to add, at position zero, a well of depth V_0 and width L . How is our story going to change? Well, we want our initial wave packet, which is up to us to choose, we want

our initial wave packet to be the same. We want to start with a Gaussian far, far, far away from the barrier. We want it to be well localized in position and well localized in momentum space, not perfectly localized, of course. It's a finite Gaussian to satisfy the uncertainty principle, but it's well localized.

So how does this story change? Well, this doesn't change at all. It's the same wave function. However, when we expand in energy eigenstates, the energy eigenstates are no longer simple plane waves. The energy eigenstates, as we know for the system, take a different form. For the square well and positive energy scattering states, the plane wave, or the energy eigenstates, which I will label by k , just because I'm going to call E is equal to $\hbar^2 k^2 / 2m$ -- the energy is a constant-- which is the k asymptotically far away from the potential.

The wave function I can write as $1/\sqrt{2\pi} e^{ikx}$ when we're on the left hand side, but being on the left hand side is equivalent to multiplying by a theta function of minus x . This is the function which is 0 when its argument is negative and 1 when its argument is positive. Plus we have the reflected term, which has an amplitude, r , which is again a function of k . This is the reflection amplitude, also known as c/a . e^{-ikx} -- again, on the left hand side-- $\theta(-x)$ plus a transmission amplitude whose norm squared is transmission probability e^{ikx} when we're on the right, $\theta(x)$.

So this is just a slightly different notation than what we usually write with left and right separated. So what we want to do now is we want to decompose our wave function in terms of the actual energy eigenstates. The way we're going to do that, and having done that, having expanded our wave function in this basis, we can determine the time evolution in the following way.

First, we expand the wave function at time 0 as an integral, dk , and I'm going to pull the $1/\sqrt{2\pi}$ out. And we have some Fourier transform, which I'm now going to call \tilde{f} because it's slightly different than the f we used before, but it's what the expansion coefficients have to be, $\tilde{f}(k)$ times this beast, $\phi(k)$. Let me actually take this product and write it out in terms of the three terms. So those three

terms are going to be $f(k)$, again $\tilde{f}(k)$, e^{-ikx} , $\theta(-x)$, plus $f(k)$ e^{ikx} , $\theta(x)$. Let's look at these terms.

Finally, we want to look at the time evolution, but we started out as a superposition of states with definite energy labeled by k , so we know the time evolution is $e^{-i(\hbar k^2/2m)t}$, $e^{-i(\hbar k^2/2m)t}$, and $kx - \omega t$. So we can immediately, from this time evolving wave function, identify these two terms as terms with a central peak moving to the right, and this, central peak moving to the left, $kx + \omega t$. Everyone cool with that? Questions?

AUDIENCE: Can you explain real quick one more time how that's the [INAUDIBLE]?

PROFESSOR: How that-- sorry, say again.

AUDIENCE: How the top equation [INAUDIBLE].

PROFESSOR: How the top equation led to this, or just where this came from?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Good. This is just a notational thing. Usually when I say $\phi(k)$ is equal to, on the left, e^{-ikx} with some overall amplitude. Let's say $\frac{1}{2\pi} e^{-ikx + c}$ over a e^{-ikx} on the left, and because this is the reflected wave, I'm just going to call this R . And then on the right, we have e^{ikx} . There's only a wave travelling to the right and the coefficient is the transmission amplitude.

So this is what we normally write, but then I'm using the so-called theta function. Theta of x is defined as 0 when x is less than 0 and 1 when x is greater than 0. This is a function 1. Using theta function allows me to write this thing as a single function without having to go around with lots of terms. Is that cool? Great.

AUDIENCE: Are we just thinking about one step here or the whole well, because always, we should have something [INAUDIBLE].

PROFESSOR: Fantastic. Excellent, excellent. Thank you so much. What I want to do is I want to think of the wave function. This is a good description when the particle is far, far away, and this is a good description when the particle is not in the potential. Sorry. Thank you. I totally glossed over this step. I want to imagine this as a potential where all of the matching is implemented at x equals 0, so when I write it in this fashion.

Another equivalent way to think about this is this is a good description of the wave function when we're not inside the well. So for the purposes of the rest of the analysis that I'm going to do, this is exactly what the form of the wave function is when we're not inside the well, and then let's just not use this to ask about questions inside the well. When I write it in this θ form, or for that matter, when I write it in this form, this is not the form. What I mean is left of the well and right of the well, and inside-- t e to the ikx -- inside, it's doing something else, but we don't want to ask questions about it. That's just going to simplify my life. Yeah?

AUDIENCE: And is the lowercase t then just the square root of capital T ?

PROFESSOR: It's the square root of capital T but you've got to be careful because there can be a phase. Remember, this is the amplitude, and what it really is is this reflection is B over A , and these guys are complex numbers. And it's true that B over A norm squared is the transmission probability, but B over A has a phase, and that quantity I'm going to call t , and we'll interpret that in more detail in a few minutes.

So it's of this form. I just want to look at each of these three terms. In particular, I want to focus on them at time 0. So at t equals 0, what does this look like? Well, that first term is integral dk of f of k e to the ikx minus ωt , θ of minus x . Notice that θ of minus x is independent of k . It's independent of the integral, so this is just a function times θ of minus x .

And this function was constructed to give us the initial Gaussian. And this was at t equals 0. So this is just our Gaussian at position x_0 centered around value k_0 at time equals 0, θ of minus x . Sorry, at position x_0 . All this function is, this is the Fourier transform of our Gaussian and we're undoing the Fourier transform. This is

just giving us our Gaussian back. And as long as the particle is far away from the well-- so here's the well and here's our wave packet-- that theta function is totally irrelevant because the Gaussian makes it 0 away from the center of the Gaussian anyway.

And so just as a quick question, if we look at this is a function of t so we put back in the minus ωt , so if we put back in the time dependence, this is a wave that's moving to the right, and to be more precise about that, what does it mean to say it's a wave moving to the right? Well, this is an envelope on this set of plain waves, and the envelope, by construction, was well localized around position x but it was also well localized in momentum, and in particular, the Fourier transform is well localized around the momentum k_0 .

And so using the method of stationary phase, or just asking, where is the phase constant stationary, we get that the central peak of the wave function satisfies the equation d/dk of $kx - \omega t$. If you're not familiar with stationary phase, let the recitation instructors know and they will discuss it for you. The points of stationary phase of this superposition of this wave packet lie at the position d/dk of $kx - \omega t$ evaluated at the peak, k_0 , of the distribution.

But d/dk of $kx - \omega t$ is, for the first term x , and for the second term, d/dk of $-\omega$. Well, we know that ω , we're in the free particle regime. ω , we all know what it is. It's $\hbar k^2 / 2m$. We take the derivative with respect to k . We get $\hbar k / m$. The 2's cancel, so $x - \hbar k / m t$.

And a place where this phase is 0, where the phase is stationary, moves over time as x is equal to $\hbar k / m t$. But $\hbar k / m$, that's the momentum. p / m , that's the classical velocity. So this is $v_0 t$, the velocity associated with that momentum. So this is good. We've done the right job of setting up our wave packet. We built a Gaussian that was far away that was moving in with a fixed velocity towards the barrier, and now we want to know what happens after it collides off the barrier. Cool?

So what we really want to ask is at late times, what does the wave function look

like? Again, we're focusing on the first term. The position of this wave packet at late times, a positive t , is positive, and when it's positive, then this theta function kills its contribution to the overall wave function. This theta is now theta of minus a positive number and this Gaussian is gone.

What is it replaced by? Well, these two terms aren't necessarily 0. In particular, this one is moving to the left. So as time goes forward, x is moving further and further to the left, and so this theta function starts turning on. And similarly, as x goes positive, this theta function starts turning on as well. Let's focus on this third term, the transmitted term.

Let's focus on this third term. In particular, that term looks like $\int dk \text{ over } \sqrt{2\pi} f$ times the transmission amplitude times e to the $ikx - \omega t$. And there's an overall theta of x outside, but for late times where the center of the wave packet where the transmitted wave packet should be positive, this theta is just going to be 1 so we can safely ignore it. It's just saying we're far off to the right.

And now I want to do one last thing. This was an overall envelope. This t was our scattering amplitude, and I want to write it in the following form. I want to write it as $\sqrt{T} e^{-i\phi}$ where ϕ is k . So what this is saying is that indeed, as was pointed out earlier, the norm squared of this coefficient T is the transmission probability, but it has a phase. And what I want to know is what does this phase mean? What information is contained in this phase? And that's what we're about to find.

So let's put that in. Here we have \sqrt{T} and $-i\phi$ where ω and ϕ are both functions of k because the transmission amplitudes depend on the momentum, or the energy. And now I again want to know, how does this wave packet move? If I look at this wave packet, how does it move en masse? As a group of waves, how does this wave packet move? In particular, with what velocity?

I'll again make an argument by stationary phase. I'll look at a point of phase equals 0 and ask how it moves over time. And the point of stationary phase is again given by d/dk of the phase $kx - \omega(k)t - \phi(k)$ is equal to 0 evaluated at

k_0 , which is where our envelope is sharply peaked. This expression is equal to-- this is, again, x minus $d\omega/dk$. That's the classical velocity, v_0 -- minus $d\phi/dk$.

But just as a note, $d\phi/dk$ is equal to $d\omega/dk$, $d\phi/d\omega$, but this is equal to-- this is just the chain rule. $d\omega/dk$, that's the classical velocity. $d\phi/d\omega$, that's $d\phi/dE$ times \hbar . I just multiplied by \hbar on the top and bottom. So this is equal to x minus $v_0 t$ plus $\hbar d\phi/dE$.

So first off, let's just make sure that the units make sense. That's a length, that's a velocity, so this had better have units of time. Time, that's good. So \hbar times $d\phi/dE$. Well, phase is dimensionless, energy is units of energy, \hbar is energy times time, so this dimensionally works out. So this is 0. So the claim is the point of stationary phase has this derivative equal to 0, so setting this equal to 0 tells us that the peak of the wave function moves according to this equation.

So this is really satisfying. This should be really satisfying for a couple of reasons. First off, it tells you that the peak of the transmitted wave packet, not just a plain wave, the peak of the actual wave packet, well localized, moves with overall velocity v_0 , the constant velocity that we started with, and that's good. If it was moving with some other velocity, we would have lost energy somehow. That would be not so sensible.

So this wave packet is moving with an overall velocity v_0 . However, it doesn't move along just as the original wave packet's peak had. It moves as if shifted in time. So the phase, and more to the point, the gradient of the phase with respect to energy, the rate of change with energy of the phase times \hbar is giving us a shift in the time of where the wave packet is. What does that mean? Let's be precise about this.

Let's interpret this. Here's the interpretation I want to give you. Consider classically the system. Classically, we have an object with some energy, and it rolls along and it finds a potential well, and what happens when it gets into the potential well? It speeds up. because it's got a lot more energy relative to the potential. So it goes much faster in here and it gets to the other side and it slows down again.

So if I had taken a particle with velocity-- let's call this position 0 and let's say it gets to this wall at time 0, so it's moving with x is equal to $v_0 t$, if there had been no barrier there, then at subsequent times, it would get out here in a time that distance over v_0 , right? However, imagine this well was extraordinarily deep. If this well were extraordinarily deep, what would happen? Well basically, in here, its velocity is arbitrarily large, and it would just immediately jump across this well.

This would be a perfectly good description of the motion before it gets to the well, but after it leaves the well, the position is going to be $v_0 t$ plus-- well, what's the time shift? The time shift is the time that we didn't need to cross this gap. And how much would that time have been? Well, it's the distance divided by the velocity. That's the time we didn't need. So t plus-- it moves as if it's at a later time-- t plus L over v_0 . Cool?

So if we had a really deep well and we watched a particle, we would watch it move x_0 , x $v_0 t$, $v_0 t$, $v_0 t$, $v_0 t$ plus L over v , $v_0 t$ plus L over v . So it's the time that we made up by being deep in the well. So there's a classical picture because it goes faster inside. So comparing these, here we've done a calculation of the time shift due to the quantum particle quantum mechanically transiting the potential well.

So let's compare these. This says the classical prediction is the time it took, Δt , classical is equal to L over v_0 , and the question is, is this the same? One way to phrase this question is, is it the same as $\hbar d \phi / dE$ evaluated at k_0 ?

From our results last time, for the amplitude c over a , we get that ϕ is equal to-- which is just minus the argument, or the phase, of c over a , or of the transmission amplitude little t -- ϕ turns out to be equal to $k_2 L$ minus \arctan of k_1^2 plus k_2^2 squared over $2 k_1 k_2$, \tan of $k_2 L$. I look at this and it doesn't tell me all that much. It's a little bit bewildering, so let's unpack this. What we really want to know is, is this close to the classical result?

Here's a quick way to check. We know this expression is going to simplify near resonance where the sine vanishes, so let's look, just for simplicity, near the

resonance. And in particular, let's look near the resonance $k^2 L$ is equal to $n \pi$. Then it turns out that a quick calculation gives you that $\hbar d \phi / dE$ at this value of k , at that value of the energy, goes as L over $2 v_0$ times 1 plus the energy over the depth of the well, v_0 .

Now remember, the classical approximation was L over v_0 . We just did this very quickly. We did it assuming an arbitrarily deep well. So v_0 is arbitrarily larger in magnitude than E , so this term is negligible. Where we should compare that very simple, naive classical result is here, L over $2 v_0$. And what we see is that the quantum mechanical result gives a time shift which is down by a factor of 2 .

So what's going on? Well, apparently, the things slow down inside. The time that it took us to cross was greater than you would have naively guessed by making it arbitrarily deep. And we can make that a little more sharp by plotting, as a function of E over v_0 , the actual phase shift.

If do a better job than saying it's infinitely deep, the classical prediction looks something like this, and this is for Δt , the time shift, classical. When you look at the correct quantum mechanical result, here's what you find, where the difference is a factor of 2 , $1/2$ the height down, and again, $1/2$ the height down. So this is that factor of 2 downstairs.

So the wave packet goes actually a little bit faster than the classical prediction would guess except near resonance, and these are at the resonant values of the momentum. At the resonant values of the momentum, it takes much longer to get across. Instead of going a little bit faster than the classical result, it goes a factor of 2 slower than the classical result.

And so now I ask you the question, why is it going more slowly? Why does it spend so much more time inside the well quantum mechanically than it would have classically? Why is the particle effectively taking so much longer to transit the well near resonance?

AUDIENCE:

Because it can reflect and it can keep going and a classical particle is not going to

do that.

PROFESSOR: Yeah, exactly. So the classical particle just goes across. The quantum mechanical particle has a superposition of contributions to its amplitude where it transits-- transit, bounce, bounce, transit, transit, bounce, bounce, bounce, bounce, transit. And now you can ask, how much time was spent by each of those imaginary particles imaginarily moving across? And if you're careful about how you set up that question, you can recover this factor of 2, which is kind of beautiful.

But the important thing here is when you're hitting resonance, the multiple scattering processes are important. They're not canceling out. They're not at random phase. They're not interfering destructively with each other. They're interfering constructively, and you get perfect transmission precisely because of the constructive interference of an infinite number of contributions to the quantum mechanical amplitude. And this is, again, we're seeing the same thing in this annoying slow down.

This tells us another thing, though, which is that the scattering phase, the phase in the transmission amplitude, contains an awful lot of the physics of the system. It's telling us about how long it takes for the wave packet to transit across the potential, effectively. Yeah?

AUDIENCE: What's the vertical axis being used for?

PROFESSOR: Sorry. The vertical axis here is the shift in the time due to the fact that it went across this well and went a little bit faster inside. So empirically, what it means is when you get out very far away and you watch the motion of the wave packet, and you ask, how long has it been since it got to the barrier in the first place, it took less time than you would have guessed by knowing that its velocity is v_0 , and the amount of time less is this much time. That answer your question? Good. Yeah?

AUDIENCE: Why is each of the amplitudes $1/2$? Why is the first one, and it bounces around twice.

PROFESSOR: When we compare the classical amplitude and the limit that v_0 goes to infinity? The

comparison is just a factor of 2. It's more complicated out here. In the limit that v_0 is large, this is exactly $1/2$.

AUDIENCE: [INAUDIBLE].

PROFESSOR: The resonances are leading to this extra factor of $1/2$. I have to say I don't remember exactly whether, as you include the sub-leading terms of 1 over v_0 , whether it stays $1/2$ or whether it doesn't, but in the limit that v_0 is large, it remains either close to $1/2$ or exactly $1/2$, I just don't remember exactly. The important thing is that there's a sharp dip. It takes much longer to transit, and so you get less bonus time, as it were. You've gained less time in the quantum mechanical model than the classical model. And when you do the experiment, you get the quantum result. That's the crucial point. Other questions?

So the phase contains an awful lot of the physics. So I want to generalize this whole story in a very particular way, this way of reorganizing the scattering in 1D. What we're doing right now is we're studying scattering problems in one dimension, but we live in three dimensions. The story is going to be more complicated in three dimensions. It's going to be more complicated in two dimensions, but the basic ideas are all the same. It's just the details are going to be different.

And one thing that turns out to be very useful in organizing scattering, both in one dimension and in three dimensions, is something called the scattering matrix. I'm going to talk about that now. In three dimensions, it's essential, but even in one dimension, where it's usually not used, it's a very powerful way to organize our knowledge of the system as encoded by the scattering data.

So here's the basic idea. As we discussed before, what we really want to do in the ideal world is take some unknown potential in some bounded region, some region, and outside, we have the potential is constant. [INAUDIBLE] my bad artistic skills. So potential is constant out here. And the potential could be some horrible thing in here that we don't happen to know, and we want to read off of the scattering process, we want to be able to deduce something about the potential.

So for example, we can deduce the energy by looking at the position of the barriers, and we can disentangle the position of the energy and the depth and the width by looking at the phase shift, by looking at the time delay. So we can deduce all the parameters of our potential by looking at the resonance points and the phase shift. I want to do this more generally for general potential. And to set that up, we need to be a bit more general than we've been.

So in general, if we solve this potential, as we talked about before, we have A and B , e^{ikx} to the minus ikx out here, and out here, we have $C e^{ikx}$ and $D e^{-ikx}$. And I'm not going to ask what happens inside. Now we can do, as discussed, two kinds of scattering experiments. We can send things in from the left, in which case A is nonzero, and then things can either transmit or reflect, but nothing's going to come in from infinity, so D is 0. Or we can do the same in reverse. Send things in from here, and that corresponds to A is 0, nothing coming in this way but $D \neq 0$ and 0.

And more generally, we can ask the following question. Suppose I send some amount of stuff in from the left and I send some amount of stuff in from the right, D . Then that will tell me how much stuff will be going out to the right and how much stuff will be going out to the left. If you tell me how much is coming in, I will tell you how much is coming out, B , C .

So if you could solve this problem, the answer is just some parilinear relations between these, and we can write this as a matrix, which I will S_{11} , S_{12} , S_{21} , S_{22} . What this matrix is doing is it takes the amplitude you're sending in from the left to right and tells you the amplitude coming out to the left or to the right. Yes?

AUDIENCE: How do we know this relation is linear?

PROFESSOR: If you double the amount of stuff coming in, then you must double the amount of stuff going out or probability is not conserved. Also, we've derived these relations. You know how the relations work. The relations work by satisfying a series of linear equations between the various coefficients such that you have continuity and differentiability at all the matching points. But the crucial thing of linearity is

probability is conserved and time evolution is linear. Other questions?

So this is just some stupid matrix, and we call it, not surprisingly, the S matrix in all of its majesty. The basic idea is this. For scattering problems, if someone tells you the S matrix, and in particular, if someone tells you how all the coefficients of the S matrix vary with energy, then you've completely solved any scattering problem. You tell me what A and D are. Great. I'll tell you exactly what B and C are mode by mode, and I can do this for superposition. So this allows you to completely solve any scattering problem in quantum mechanics once you know it. So it suffices to know S to solve all scattering problems.

I want to now spend just a little bit of time thinking about what properties the S matrix and its components must satisfy. What properties must the S matrix satisfy in order to jibe well with the rest of the rules of quantum mechanics? I'm not going to study any particular system. I just want to ask general questions.

So the first thing that must be true is that stuff doesn't disappear. Disapparate is probably the appropriate. Stuff doesn't leak out of the world, so whatever goes in must come out. What that means is norm of A squared plus norm of D squared, which is the probability density in and probability density out, must be equal to B squared plus C squared. Everyone agree with that?

AUDIENCE: Why can't stuff stay in the potential?

PROFESSOR: Why can't stuff stay in the potential? That's a good. So if we're looking at fixed energy eigenstates, we know that we're in a stationary state, so whatever the amplitude going into the middle is, it must also be coming out of the middle. There's another way to say this, which is let's think about it not in terms of individual energy eigenstates. Let's think about it in terms of wave packets. So if we take a wave packet of stuff and we send in that wave packet, it has some momentum, right?

This is going to get delicate and technical. Let me just stick with the first statement, which is that if stuff is going in, then it has to also be coming out by the fact that this is an energy eigenstate. The overall probability distribution is not changing in time. If

stuff went in and it didn't come out, that would mean it's staying there. That would mean that the probability density is changing in time. That's not what happens in an energy eigenstate. The probability distribution is time independent. Everyone cool with that?

So let's think, though, about what this is. I can write this in the following nice way. I can write this as A complex conjugate D complex conjugate, AD . And on the right hand side, this is equal to B complex conjugate, C complex conjugate, BC . I have done nothing other than write this out in some suggested form.

But B and C are equal to the S matrix, so BC is the S matrix times A , and B complex conjugate, C complex conjugate is the transposed complex conjugate. So this is equal to A complex conjugate, D complex conjugate, S transpose complex conjugate, also known as adjoint, and this is S on AD . Yeah?

But this has to be equal to this for any A and D . So what must be true of S dagger S as a matrix? It's got to be the identity as a matrix in order for this to be true for all A and D . Ah. That's cool. Stuff doesn't disappear. S is a unitary matrix. So S is a unitary matrix. Its inverse is its adjoint. You'll study this in a little more detail on the problem set. You studied the definition of unitary on the last problem set.

So that's the first thing about S and it turns out to be completely general. Anytime, whether you're in one dimension or two dimensions or three, if you send stuff in, it should not get stuck. It should come out. And when it does come out, the statement that it comes out for energy eigenstates is the statement that S is a unitary matrix. Questions on that.

So a consequence of that is that the eigenvalues of S are phases, pure phases. So I can write S -- I'll write them as S_1 is equal to $e^{i\phi_1}$ and S_2 is equal to $e^{i\phi_2}$. So the statement that S is a unitary matrix leads to constraints on the coefficients, and you're going to derive these on your problem set. I'm just going to list them now.

The first is that the magnitude of S_{11} is equal to the magnitude of S_{22} . The

magnitude of S_{12} is equal to the magnitude of S_{21} . More importantly, S_{12} norm squared plus S_{11} squared is equal to 1. And finally, S_{11} , S_{12} complex conjugate, plus S_{21} , S_{22} complex conjugate is equal to 0.

So what do these mean? What are these conditions telling us? They're telling us, of course, they're a consequence of conservational probability, but they have another meaning. To get that other meaning, let's look at the definition of the transmission amplitudes. In particular, consider the case that we send stuff in from the left and nothing in from the right. That corresponds to D equals 0.

When D equals 0, what does this tell us? It tells us that B is equal to-- and A equals 1 for normalization. B is equal to-- well, D is equal to 0, so it's just $S_{11} A$, so B over A is S_{11} . And similarly, C over A is S_{21} . But C over A is the thing that we've been calling the transmission amplitude, little t , and this is the reflection amplitude, little r .

So this is the reflection amplitude, if we send in stuff and it bounces off, reflects to the left, and this is the transmission amplitude for transmitting to the right. Everyone cool with that? This is reflection to the left, this is transmission to the right. By the same token, this is going to be transmission to the left and reflection to the right.

So now let's look at these conditions. S_{11} is reflection and this is going to be reflection. This says that the reflection to the left is equal to the reflection to the right in magnitude. Before, what we saw was for the simple step, the reflection amplitude was equal from left to right, not just the magnitude, but the actual value was equal from the left and the right. It was a little bit of a cheat because they were real.

And we saw that that was a consequence of just being the step. We didn't know anything more about it. But now we see that on general grounds, on conservation of probability grounds, the magnitude of the reflection to the left and to the right for any potential had better be equal. And similarly, the magnitude of the transmission for the left and the transmission to the right had better be equal, all other things being-- if you're sending in from the left and then transmitting to the left, or sending in from the right and transmitting to the right.

And what does this one tell us? Well, S_{12} and S_{11} , that tells us that little t squared, which is the total probability to transmit, plus little r squared is the total probability to reflect, is equal to 1, and we saw this last time, too. This was the earlier definition of nothing gets stuck. And this one you'll study on your problem set. It's a little more subtle. Questions? Yeah?

AUDIENCE: Can you explain one more time why is S unitary? How did you get that?

PROFESSOR: So the way we got S was unitary is first off, this is just the definition of S . S is the matrix that, for any energy, relates A and D , the ingoing amplitudes, to the outgoing amplitudes B and C . Just the definition. Meanwhile, I claim that stuff doesn't go away, nothing gets stuck, nothing disappears, so the total probability density of stuff going in must be equal to the total probability density of stuff going out.

The probability of stuff going in can be expressed as this row vector times this column vector, and on the out, this row vector times this column vector, and then we use the definition of the S matrix. This column vector is equal to S times this row vector. BC is equal to S times AD . And when we take the transposed complex conjugate, I get AD transposed complex conjugate, S transposed complex conjugate, but that's the S adjoint. But in order for this to be true for any vectors A and D , it must be that $S^\dagger S$ is the identity, but that's the definition of a unitary matrix. Cool? Others.

You're going to prove a variety of things on the problem set about the scattering matrix and its coefficients, but I want to show you two properties of it. The first is reasonably tame, and it'll make a little sharper the step result we got earlier that the reflection in both directions off the step potential was in fact exactly the same.

Suppose our system is time reversal invariant. So if t goes to minus t , nothing changes. This would not be true, for example, if we had electric currents in our system because as we take t to minus t , then the current reverses. So if the current shows up in the potential, or if a magnetic field due to a current shows up in the potential energy, then as we change t to minus t , we change the direction of the current, we change the direction of the magnetic field.

In simple systems where we have time reversal invariance, for example, electrostatics, but not, for example, magnetostatics, suppose we have time reversal invariance, then what you've shown on a previous problem set is that ψ is a solution, then ψ^* , ψ complex conjugate is also a solution. And using these, what you'll see what you can show-- and I'm not going to go through the steps for this-- well, that's easy. If we do the time reversal, the wave function, looking on the left or on the right, so comparing these guys, what changes? Under time reversal, we get a solution. Given this solution, we have another solution, $A \rightarrow A^*$ to the minus ikx , $B \rightarrow B^*$ to the plus ikx . Minus star, star, plus.

So we can run exactly the same game but now with this amplitude. And when you put the conditions together, which you'll do on the problem set, another way to say this is the same solution with $k \rightarrow -k$ and with A and B replaced by their complex conjugates and C and D replaced by each other's complex conjugates. Then this implies that it must be true that A and D , which are now the outgoing guys because we've time reversed, is equal to S_{11}^* and I'll write this out explicitly-- S_{11}^* , S_{12}^* , S_{21} , S_{22} , B^* , and C^* .

So these together give you that. Therefore, $S^* S = 1$. If $S^* S = 1$, then $S^{-1} = S^*$, just putting this on the right. So $S^* = S^{-1} = S^\dagger$ because S is also unitary. So time reversal invariance implies, for example, that $S_{12}^* = S_{21}$, or $S_{12} = S_{21}^*$.

This is what I wanted to write here. Gives us that $S_{12} = S_{21}^*$. And in particular, this tells us that S_{21} is equal to S_{12}^* . The off diagonal terms are equal, not just in magnitude, which was insured by unitary, but if, in addition to being a unitary system, which of course, it should be, if in addition it's time reversal invariant, then we see that the off diagonal terms are equal not just in magnitude but also in phase.

And as you know, the phase is important. The phase contains physics. It tells you about time delays and shifts in the scattering process. So the phases are the same.

That statement is not a trivial one. It contains physics. So when the system is time reversal invariant, the phases as well as the amplitudes are the same. And you'll derive a series of related conditions or consequences for the S matrix from various properties of the system, for example, parity, if you could have a symmetric potential.

But now in the last few minutes, I just want to tell you a really lovely thing. So it should be pretty clear at this point that all the information about scattering is contained in the S matrix and its dependence on the energy. If you know what the incident amplitudes are, you know what the outgoing amplitudes are, and that's cool because you can measure this.

You can take a potential, you can literally just send in a beam of particles, and you can ask, how likely are they to get out. And more importantly, if I build a wave packet, on average, what's the time delay or acceleration of the transmitted wave packet? And that way, I can measure the phase as well. I can measure both the transmission probabilities and the phases, or at least the gradient of the phase with energy. Go ahead.

AUDIENCE: Is there a special condition that we can [pose ?] to see the resonance?

PROFESSOR: Excellent question. Hold onto your question for a second. There's an enormous amount of the physics of scattering contained in the S matrix, and you can measure the S matrix, and you can measure its dependence on the energy. You can measure the coefficients S_{12} and S_{22} , their phases and their amplitudes, as a function of energy, and you can plot them.

Here's what I want to convince you of. If you plot those and look at how the functions behave as functions of energy and ask, how do those functions extend to negative energy by just drawing the line, continuing the lines, you can derive the energy of any bound states in the system, too. Knowledge of the scattering is enough to determine the bound state energies of a system, and let me show you that. And this is one of the coolest things in quantum mechanics.

Here's how this works. We have, from the definition of the S matrix, that BC is equal to the S matrix on AD, where the wave function-- let me just put this back in the original form-- is $Ce^{ikx} + Ae^{-ikx}$, e to the ikx , e to the $-ikx$, and e to the ikx , e to the $-ikx$. So that's the definition of the S matrix. The S matrix, at a given energy e , is a coefficient relation matrix between the ingoing and outgoing, or, more to the point, A and D. And in all of this, I've assumed that the energy was positive, that the k_1 and k_2 are positive and real.

But now let's ask the question, what would have happened if, in the whole process, I had taken the energy less than 0? If the energy were less than 0, instead of k , k would be replaced by $i\alpha$. Let's think about what that does. If k is $i\alpha$, this is e to the ik is $e^{-\alpha x}$ and e^{-ikx} is $e^{+\alpha x}$. Similarly, ik times i , that gives me a $-\alpha$ and this gives me a $+\alpha$. Yeah? So as equations, they're the same equations with k replaced by $i\alpha$.

And now what must be true for these states to be normalizable? What must be true, for example, of A? A must be 0 because at minus infinity, there's divergence. Not normalizable. So in order to be at bound state, in order to have a physical state, A must be 0. What about D? Same reason. It's got to be 0 at positive infinity. These guys are convergent, so C and B can be non-zero.

So now here's my question. We know that these relations must be true because all these relations are encoding is how a solution here matches to a solution here through a potential in between with continuity of the derivative and anything else that's true of that potential inside. All that S is doing from that point of view is telling me how these coefficients match onto these coefficients. Yes?

Now what, for a bound state-- if we have e less than 0, what must be true? It must be true that AD is equal to 0, and in particular, 00. So what are B and C? Well, a matrix times 0 is equal to--

AUDIENCE: 0.

PROFESSOR: Unless?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Unless the matrix itself is diverging, and then you have to be more careful, but let's be naive for the moment. If A is 00, then in order for B and C to be non-zero, S must have a pole. S must go like 1 over 0 . S must diverge at some special value of the energy.

Well, that's easy. That tells you that if you look at any particular coefficient in S , any of the matrix elements of S , the numerator can you whatever you want, some finite number, but the denominator had better be?

AUDIENCE: 0.

PROFESSOR: 0. So let's look at the denominator. If I compute S_{21} -- actually, let me do this over here. No, it's all filled. Let's do it over here. If I look at S_{21} for the potential well, scattering off the potential well that we looked at at the beginning of today's lecture, this guy, and now I'm going to look at S_{21} , one of the coefficients of this guy, also known as the scattering amplitude t for the well. This is equal to-- and it's a godawful expression-- $2 k_1 k_2 e^{-i k_2 L} / (2 k_1 k_2 \cos(k' L) - i k_1^2 + k_2^2 \sin^2(k_2 L))$. This is some horrible thing.

But now I ask the condition, when does this have a pole? When the energy has continued to be negative, for what values does this have a pole or does the denominator have a 0? And the answer is if you take this and you massage the equation, this is equal to 0 a little bit, you get the following expression, $k_2 L \tan(k_2 L) = k_1 L$. This is the condition for the bound state energies of the square well, and we computed it using knowledge only of the scattering states.

If you took particles and a square well and roll the particles across the square well potential and measure it as a function of energy, the scattering amplitude, the transmission amplitude, and in particular S_{21} , an element of the S matrix, and you plotted it as a function of energy, and then you approximated that by a function of

energy that satisfies the basic properties of unitarity, what you would find is that when you then extend that function in mathematica to minus a particular value of the energy, the denominator diverges at that energy. You know that there will be a bound state. And so from scattering, you've determined the existence of a bound state. This is how we find an awful lot of the particles that we actually deduce must exist in the real world. We'll pick up next time.

[APPLAUSE]