8.04: Quantum Mechanics
Massachusetts Institute of Technology

Professor Allan Adams
Tuesday Feb 12

Problem Set 2

Due Thursday Feb 21 at 11.00AM

Assigned Reading:

E&R	3.(all)	5.(1,3,4,6)
Li.	2.(5-8)	3.(1-3)
Ga.	2-(all except 4)	
Sh.	3, 4	

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1. **(10 points) Wave-riding Mechanics**

   (a) What is the difference between phase velocity and group velocity? How do they differ in gravity waves in the open ocean?

   (b) Why do waves break on a beach? Waves seem to get taller as they approach a beach – is this accurate, or an optical illusion?

   (c) A surfer is waiting for a wavepacket of ocean waves which is approaching a beach. (For concreteness, let it be an infinite linear beach with beautiful beige sand, constant slope, warm air and little surf shacks and ceviche bars every few kilometers.) To the surfer's frustration, the first few waves appear to dissolve before arriving. Is this accurate, or an illusion? Use the difference between wave and group velocity to explain why big waves tend to approach the beach in sets.

2. **(10 points) Why I Don't Play Craps**

   Let s be the number of spots shown by a 12-sided die thrown at random.

   (a) Calculate $\langle s \rangle$.

   (b) Calculate Δs.

   (c) Calculate $\langle s \rangle$ and Δs if you're throwing two die.

3. **(5 points) Dimensions of ψ**

   (a) What are the dimensions of $\psi(x)$?

   (b) What are the dimensions of $\tilde{\psi}(k)$?

4. **(15 points) Fourier Transforms and Expectation Values**

In 8.04 conventions, a function $f(x)$ and its fourier transform $\tilde{f}(k)$ are related by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\ e^{ikx} \tilde{f}(k) \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ e^{-ikx} f(x)\,.$$

By definition,

$$\langle x \rangle = \int dx\ \mathbb{P}(x)\, x = \int dx\ |\psi(x)|^2\, x.$$

Using the definition of expectation values of (powers of) the momentum operator,

$$\langle \hat{p}^n \rangle = \int dx\ \psi^*(x)\, (\hat{p})^n\, \psi(x)$$

the form of the momentum operator,

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

and the definition of the fourier transform,

(a) show that

$$\langle \hat{p} \rangle = \int dk\ |\tilde{\psi}(k)|^2\, \hbar k\,,$$

(b) and that

$$\langle \hat{p}^2 \rangle = \int dk\ |\tilde{\psi}(k)|^2\, (\hbar k)^2\,,$$

(c) and, in general, that

$$\langle f(\hat{p}) \rangle = \int dk\ |\tilde{\psi}(k)|^2\, f(\hbar k)\ .$$

The relation $\mathbb{P}(k) = |\tilde{\psi}(k)|^2$, as discussed in lecture, thus follows from the Born relation, $\mathbb{P}(x) = |\psi(x)|^2$.

5. **(15 points) Delta Functions**

The Dirac delta function, $\delta(x)$, is defined by the condition

$$\delta(x) = \left\{ \begin{array}{cc} 0 & x \neq 0 \\ \infty & x = 0 \end{array} \right\}$$

or, more precisely,

$$\int_a^b dx\ \delta(x)\, f(x) = \left\{ \begin{array}{cc} f(0) & 0 \in [a,b] \\ 0 & \text{otherwise} \end{array} \right\}$$

(a) Evaluate the integrals,

$$\int_{-3}^{1} dx\, \delta(x+2)\ \left(x^3 - 3x^2 + 2x - 1\right)$$
$$\int_{0}^{\infty} dx\, \delta(x-\pi)\ (\cos(3x) + 2)$$
$$\int_{-1}^{1} dx\, \delta(x-2)\, e^{|x|+3}$$

Lesson: *Delta functions are magical beasts which help you solve any integral at a glance!*

(b) Two expressions $D_1(x)$ and $D_2(x)$ involving delta functions are equal if, for any smooth function $f(x)$,

$$\int_{-\infty}^{\infty} dx\, D_1(x)\, f(x) = \int_{-\infty}^{\infty} dx\, D_2(x)\, f(x)$$

Establish the following equalities:

i. $x\,\delta(x) = 0$
ii. $\delta(-x) = \delta(x)$
iii. $\delta(c\,x) = \frac{1}{|c|}\delta(x)$
iv. $\int_{-\infty}^{\infty} dx\, \delta(a-x)\delta(x-b) = \delta(a-b)$
v. $f(x)\delta(x-a) = f(a)\delta(x-a)$

(c) Show that the following are valid representations of $\delta(x)$ (*c.f.* prob. 6 part g):

i. $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$
ii. $\delta(x) = \lim_{a\to 0}\, \delta_a(x), \qquad \delta_a(x) \equiv \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$

6. **(30 points) Qualitative Structure of Wavefunctions**

Aside: In lecture this week we looked at a number of example wavefunctions, sketched their fourier transforms, and estimated our confidence in predictions for measurements of the position and momentum of the corresponding system. In this problem you will make those arguments precise by explicitly computing everything we discussed.

Consider the following non-normalized wavefunctions[1] for a particle in one dimension, with x a *dimensionless* variable:

$$\begin{aligned} \psi_1(x) &\propto \delta(x-1) \\ \psi_2(x) &\propto \delta(x-2) \\ \psi_3(x) &\propto e^{ix} \\ \psi_4(x) &\propto e^{i2x} \\ \psi_5(x) &\propto \delta(x-1)+\delta(x-2) \\ \psi_6(x) &\propto e^{ix}+e^{i2x} \end{aligned}$$

$$\psi_7(x) = \left\{ \begin{array}{cc} N & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0 & otherwise \end{array} \right\}$$

$$\psi_8(x) = N e^{-(x-x_o)^2/a^2} e^{ik_o x}$$

For each of these wavefunctions,

(a) Sketch the wavefunction and the corresponding probability distribution.
note: *If you find yourself attempting to sketch a complex number, simplify your life and plot the real part only. Complex paper is hard to find.*
note: *If you find yourself sketching a delta function $\delta(x)$ (or its square!), you may replace the delta with a narrow gaussian, $\delta(x) \sim \delta_a(x)$ for $a \ll 1$, as you justified in Problem (5.c.ii).*

(b) Compute the Fourier transform $\tilde{\psi}(k)$.

(c) Sketch $\tilde{\psi}(k)$ and the corresponding probability distribution. How does $\mathbb{P}(k)$ behave at large k? Interpret this physically (c.f. Problem 7).

(d) Where will the particle most likely be found? What's the most likely momentum? How confident of these predictions are you? *Note: Estimate, no calculations needed.*

For $\psi_8(x)$ only:

(e) Determine the normalization constant N up to an overall phase.

(f) Calculate the expectation values $\langle x \rangle$ and $\langle p \rangle$ and the uncertainties Δx and Δp in this state. Do these values satisfy the uncertainty principle? If so, how efficiently? Comment on what this says about wavefunctions of this form.

(g) What happens to the shape of the wavefunction as $a \to 0$? What happens to $\tilde{\psi}(k)$? What happens to both in the limit $a \to \infty$? How do Δx and Δp behave in these limits? Does this make sense?

[1] See Problem 5 for practice with the δ function.

7. **(15 points) Why the Wavefunction should be Continuous**

   Consider the wavefunction

$$\psi_7(x) = \left\{ \begin{array}{ll} N & -\frac{a}{2} \le x \le \frac{a}{2} \\ 0 & otherwise \end{array} \right\}$$

   from problem 6, which is just a step function centered on the origin. On first glance this looks like a perfectly reasonable wavefunction: it is normalizable, Δx is finite, its fourier transform is simple... However, since it has *discontinuities* at $x = \pm\frac{a}{2}$, acting on it with $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ leads to *divergences*, and in particular to divergent expectation values for the momentum. That doesn't seem very physical! Let's make this precise:

   (a) Determine the normalization constant N and calculate the uncertainty Δx. How does this compare with your estimate from the problem 6?

   (b) Calculate $\langle p \rangle$ and $\langle p^2 \rangle$ by using the fourier transform you calculated in problem 6 and your results from problem 4. *Notice that your result for $\langle p^2 \rangle$ is divergent!* What must be true of the large-k behavior of $\widetilde{\psi}(k)$ to ensure that $\langle p^2 \rangle$ is finite?

   Aside: We could alternatively compute $\langle \hat{p}^2 \rangle$ via $\int dx\ \psi(x)^* \, (-\hbar^2 \frac{\partial^2}{\partial x^2})\psi(x)$. Verifying that this gives the same divergence is straightforward but involves delicate manipulations involving derivatives of delta functions – try it if you're up for a computational challenge.

   Mathematically, this divergence is due to $\psi_7(x)$ having a discontinuity: building a *discontinuous function* by superposing *continuous waves* requires superposing an infinite number of waves with comparable strength. Physically, since each wave corresponds to a definite momentum, building a discontinuous wavefunction requires arbitrarily high momentum modes. To ensure the expectation value $\langle \hat{p}^2 \rangle$ is finite, we should demand that **Physical wavefunctions must be continuous.** We will return to this point later when we talk about Energy and the Schr̈dingier equation.

8. **(Optional) Smooth Wavefunctions give finite expectation values**

As you have shown, the discontinuous wavefunction $\psi_7(x)$ gives divergent values for $\langle \hat{p}^2 \rangle$ and so seems quite unphysical. You have also shown that its fourier transform $\tilde{\psi}_7(k)$ falls off very slowly for large k. In this problem we'll explore the connection between continuity, finiteness of $\langle \hat{p}^2 \rangle$ and the large-k behavior of $\tilde{\psi}$.

Consider the *smooth* function

$$\psi_{7b}(x) = \frac{N}{2}\left[\tanh\left(\frac{x+\frac{a}{2}}{b}\right) - \tanh\left(\frac{x-\frac{a}{2}}{b}\right)\right]$$

(a) Sketch $\psi_{7b}(x)$ for various values of b to convince yourself that $\psi_{7b}(x)$ well approximates $\psi_7(x)$ as $b \to 0$. Argue that, for $b \ll a$, N is roughly independent of b. Compute N when $b \to 0$.

(b) Show that the fourier transform of $\psi_{7b}(x)$ is,

$$\tilde{\psi}_{7b}(k) = N\sqrt{\frac{\pi}{2}}\,\frac{b\,\sin(\frac{ka}{2})}{\sinh(\frac{k\pi b}{2})}\,.$$

(c) How does $|\tilde{\psi}_{7b}(k)|^2$ behave for large values of k? Compare with $|\tilde{\psi}_7(k)|^2$. What does this imply about $\langle \hat{p}^2 \rangle$?

(d) In the limit that $b \ll a$ (so that N is independent of b and you can use the value you computed above), and *without evaluating any integrals*, show that

$$\Delta p \;\propto\; \frac{1}{\sqrt{a\,b}}\,.$$

Hint: When $b \ll a$, the function $\sin^2(\frac{ax}{b})$ oscillates rapidly in x; over many periods it is thus well approximated by its average, $\sin^2(\frac{ax}{b}) \sim \frac{1}{2}$. This is a useful trick to remember.

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