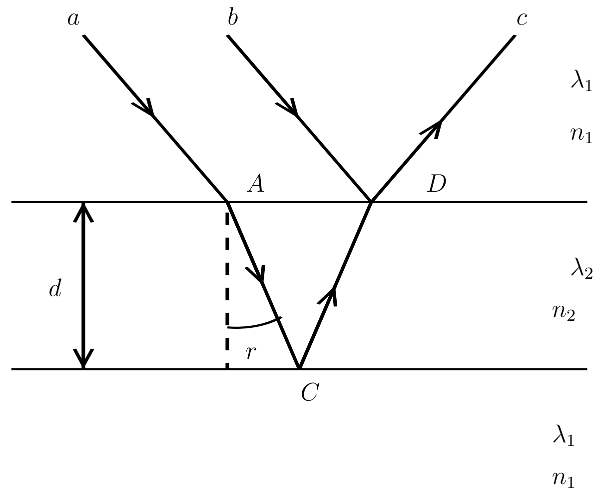
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #9 Solutions

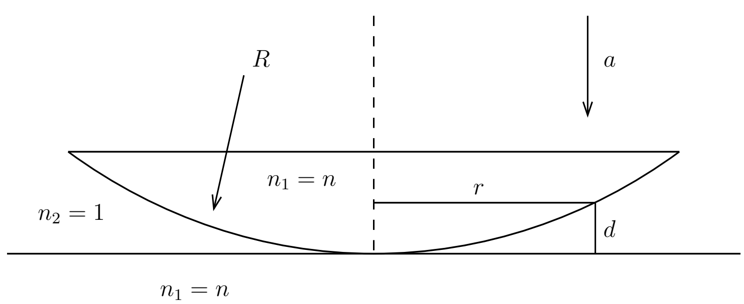
Problem 9.1: (Bekefi & Barrett 8.1)[1] Thin film interference

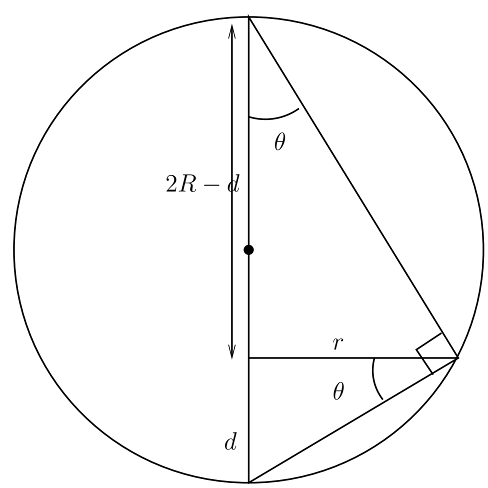


The glass is too thick to produce "thin film interference". We will therefore only concentrate on the air gap. The phase difference between a and b after 'joining' at c is $\delta = \dfrac{4\pi d}{\lambda_1}\dfrac{n_2}{n_1}\cos r + \pi$. For constructive interference, the condition is that $\delta = 2m\pi \quad (m = 1, 2, ...)$. In case of normal incidence ($\cos r = 1$), the equation can be modified to: $\lambda_{1m} = \dfrac{4dn_2}{(2m-1)n_1} \quad \lambda_2 = \dfrac{n_1}{n_2}\lambda_1 \Rightarrow \quad \lambda_2 = 4d$ for $m = 1$. $d = \lambda_2/4 = 1 \times 10^{-7}$ m where $\lambda_2 = 4 \times 10^{-7}$ m $= 100$ nm.

This answer could have been 'guessed' without this elaborate calculation. If $d = \lambda_2/4$ the extra distance traveled is $\lambda_2/2$. In addition, there will be a phase difference of π! At D, light reflects off a less dense medium (namely air) ($n_2 < n_1$) so there is no sign change in $\vec{E}$. However, at C, light reflects off the denser glass, thus changing the phase of $\vec{E}$ by π. The extra distance traveled in the air gap plus the sign change in E add up to a 2π 'change' in phase if $d = \lambda_2/4$.

Problem 9.2: (Bekefi & Barrett 8.4) Newton rings





a) From the figure on the right,

$$\sin\theta = \frac{r}{2R-d} = \frac{d}{r} \quad \Rightarrow \quad r^2 = d(2R-d) \quad \Rightarrow \quad r^2 \simeq 2dR \;\; (d^2 \ll dR) \qquad d \simeq \frac{r^2}{2R}$$

b) For rays a as shown in the left figure, the phase difference between those that travel back and

[1] The notation "Bekefi & Barrett" indicates where this problem is located in one of the textbooks used in 8.03 in 2004: Bekefi, George, and Alan H. Barrett *Electromagnetic Vibrations, Waves, and Radiation.* Cambridge, MA: MIT Press, 1977. ISBN: 9780262520478.

forth in the gap d and those that reflect off the curved surface of the glass is: $\delta = \dfrac{4\pi d}{\lambda_2} + \pi$ where $n_2 = 1, n_1 = n$ and $\lambda_1 = \lambda_2/n$. For constructive interference, the condition is that $\delta = 2m\pi$ $(m = 1, 2, ...)$. Thus $2m\pi = \pi\left(\dfrac{4d}{\lambda_2} + 1\right)$ Combining equations and replacing λ_2 with λ (it is the wavelength in air): $r_m = \left[\dfrac{(2m-1)\lambda R}{2}\right]^{1/2}$

Example: For λ=500 nm and R=10 m, some of the ring radii are:

$$r_1 = 1.58 \text{ mm} \quad r_2 = \sqrt{3} \times 1.58 \text{ mm} \quad r_{13} = \sqrt{25} \times 1.58 \text{ mm} \simeq 7.9 \text{ mm}$$

The ring spacing decreases with increasing radius r. The ratio of radius $(m+1)$ to that of the m^{th} ring is $\sqrt{(2m+1)/(2m-1)}$. For $m = 1$ the ratio is $\simeq 1.7$, for $m = 13$ it is $\simeq 1.04$.

c) For destructive interference, the condition is that $\delta = (2m+1)\pi \quad (m = 1, 2, ...)$.
$(2m+1)\pi = \pi\left(\dfrac{4d}{\lambda} + 1\right)$ and so $r_m = [m\lambda R]^{1/2}$ Substituting values of $R = 2$ m and $\lambda = 640$ nm, the values of r_m are: $r_m = 1.13\sqrt{m} \times 10^{-3}$ m.

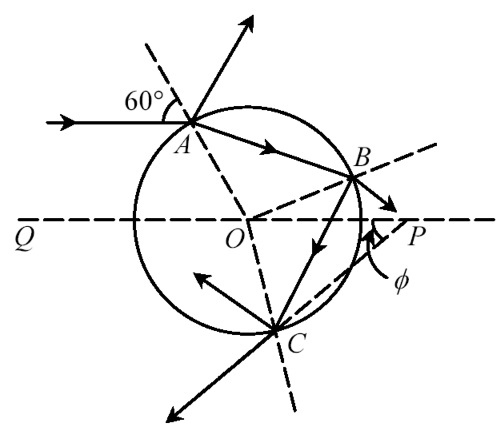
d)
$$\begin{array}{llll} r_1 = 1.13 \text{ mm} & r_2 = 1.60 \text{ mm} & \Delta r_{1,2} = r_2 - r_1 = 0.47 \text{ mm} \\ r_{25} = 5.66 \text{ mm} & r_{26} = 5.77 \text{ mm} & \Delta r_{25,26} = r_{26} - r_{25} = 0.11 \text{ mm} \end{array}$$
where r_i is the i^{th} dark ring.

Problem 9.3: Rainbows

a) Incident unpolarized light: $\parallel 0.5I_o, \perp 0.5I_o$

$\theta_1 = 60°$, $\theta_2 = 40.59°$ (Snell's Law), $n_1 = 1.0$, $n_2 = 1.331$

$$\begin{array}{ll} r_\parallel = 0.06587 & I_{r_\parallel} = 0.06587^2 \times 0.5I_0 = 0.002170I_0 \\ & I_{t_\parallel} = 0.5I_0 - 0.002170I_0 = 0.4978I_0 \\ r_\perp = -0.3381 & I_{r_\perp} = 0.3381^2 \times 0.5I_0 = 0.05715I_0 \\ & I_{t_\perp} = 0.5I_0 - 0.05715I_0 = 0.4429I_0 \end{array}$$



Degree of linear polarization of the transmitted light:

$$V = \left|\frac{I_{t_\parallel} - I_{t_\perp}}{I_{t_\parallel} + I_{t_\perp}}\right| = \left|\frac{0.498 - 0.443}{0.498 + 0.443}\right| = 0.0584,$$

5.8% linearly polarized in the parallel direction.

b) Reflection at B. Incoming radiation: $\parallel 0.443I_o \;\; \perp 0.498I_o$

$\theta_1 = \angle\, OAB = \angle\, OBA = 40.59°$, $\theta_2 = 60.00°$, $n_1 = 1.331$, $n_2 = 1.000$

$$\begin{array}{ll} r_\parallel = -0.06587 & I_{r_\parallel} = 0.06587^2 \times 0.498I_0 = 0.00216I_0 \\ r_\perp = 0.3381 & I_{r_\perp} = 0.3381^2 \times 0.443I_0 = 0.0506I_0 \end{array}$$

Polarization of the reflected light: $V = \left|\dfrac{I_{r_\parallel} - I_{r_\perp}}{I_{r_\parallel} + I_{r_\perp}}\right| = \left|\dfrac{0.00216 - 0.0506}{0.0506 + 0.00216}\right| = 0.918$ (92% polarized).

It is not surprising that the reflected light at B is so highly polarized. The Brewster angle for the transition water$\rightarrow$air is 36.9°. The angle of incidence, $\theta_1 = 40.6°$, is only $\sim 3.7°$ larger than this.

c) Radiation that arrives at C. Incoming radiation: $\parallel 0.00216I_o$, $\perp 0.0506I_o$
$\theta_1 = 40.59°, \theta_2 = 60.00°, n_1 = 1.331, n_2 = 1.000$

$$\begin{aligned} r_\parallel = -0.06587 \qquad I_{r_\parallel} &= 0.06587^2 \times 0.00216I_0 = 9.37 \times 10^{-6}I_0 \\ I_{t_\parallel} &= 0.00216I_0 - 0.00000937I_0 = 0.00215I_0 \\ r_\perp = 0.338 \qquad I_{r_\perp} &= 0.3381^2 \times 0.0506I_0 = 0.00578I_0 \\ I_{t_\perp} &= 0.0506I_0 - 0.00578I_0 = 0.04482I_0 \end{aligned}$$

Polarization of the transmitted light: $V = \left|\frac{I_{t_\parallel} - I_{t_\perp}}{I_{t_\parallel} + I_{t_\perp}}\right| = \left|\frac{0.00215 - 0.04482}{0.04482 + 0.00215}\right| = 0.9085.$

In conclusion: The intensity of light emerging into the air at C is 4.7% ($\perp$ 4.48% , $\parallel$ 0.22%) of I_0, i.e. 91% linearly polarized in the $\perp$ direction.

d) Angle of incidence and refraction are θ_1 and θ_2, respectively. $\angle AOB = \angle BOC = 180° - 2\theta_2$
$\Rightarrow \angle AOC = 4\theta_2$. $\angle QOC = 4\theta_2 - \theta_1 \Rightarrow \angle POC = 180° - 4\theta_2 + \theta_1$. $\angle OCP = \theta_1$
$\Rightarrow \phi = 180° - \angle POC - \angle OCP = 180° - 180° + 4\theta_2 - 2\theta_1$. So, the end result is $\phi = 4\theta_2 - 2\theta_1$

e) Red Light $n = 1.331$ $\theta_1 = 60°$ $\theta_2 = 40.59°$ $\qquad \phi_{red} = 4\theta_2 - 2\theta_1 = 42.4°$

Blue/violet Light $n = 1.343$ $\theta_1 = 60°$ $\theta_2 = 40.15°$ $\qquad \phi_{red} = 4\theta_2 - 2\theta_1 = 40.6°$

f) $\phi = 4\theta_2 - 2\theta_1 \qquad \frac{d\phi}{d\theta_1} = 4\frac{d\theta_2}{d\theta_1} - 2 = 0 \quad \Rightarrow \quad \frac{d\theta_2}{d\theta_1} = \frac{1}{2}$

$$\sin\theta_1 = n\sin\theta_2 \qquad \cos\theta_1 d\theta_1 = n\cos\theta_2 d\theta_2 \quad \Rightarrow \quad \frac{d\theta_2}{d\theta_1} = \frac{\cos\theta_1}{n\cos\theta_2} = \frac{1}{2}$$

$$\begin{aligned} \cos\theta_2 = \frac{2}{n}\cos\theta_1 = (1-\sin^2\theta_2)^{1/2} &= (1 - \frac{1}{n^2}\sin^2\theta_1)^{1/2} = (1 - \frac{1}{n^2} + \frac{1}{n^2}\cos^2\theta_1)^{1/2} \\ \frac{4}{n^2}\cos^2\theta_1 = 1 - \frac{1}{n^2} + \frac{1}{n^2}\cos^2\theta_1 &\Rightarrow \frac{3}{n^2}\cos^2\theta_1 = 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} \\ \cos^2\theta_1 &= \frac{n^2-1}{3} \end{aligned}$$

g) Red Light: $\cos^2\theta_1 = (1.331^2 - 1)/3 = 0.257$ $\quad \theta_1 = 59.5°$ $\quad \theta_2 = 40.3°$ $\phi_{max} = 4\theta_2 - 2\theta_1 = 42.4°$

Blue/violet Light: $\cos^2\theta_1 = (1.343^2 - 1)/3 = 0.2678$ $\theta_1 = 58.8°$ $\theta_2 = 39.6°$ $\phi_{max} = 4\theta_2 - 2\theta_1 = 40.6°$

The width of the visible-color region of the rainbow is therefore about $42.4° - 40.6° + 0.5° = 2.3°$ The 0.5° is added due to the fact that the sun has a diameter of 0.5°. Thus, the width of the rainbow is about 5–6% of its radius.

h) $n = 1.5 \qquad \theta_1 = \sin^{-1}[1 - (n^2-1)/3]^{1/2} = 49.8° \qquad \theta_2 = 30.6° \qquad \phi_{max} = 4\theta_2 - 2\theta_1 = 22.8°$
Notice: the radius of the glass bow is only about half that of the rainbow! The glass bow is also highly polarized as the Brewster angle (glass to air) is $\sim 33.7°$; the angle of incidence at the reflection at B is only $\sim 3°$ smaller.

Problem 9.4: (Bekefi & Barrett 8.5) Superposition of N oscillators

a) Let the points after addition of each successive phasor be A, B, C, D and F.

A (3, 0),

B (3+3cos20°, 3sin20°),

C $(B_x + 3\cos 40°,\ B_y + 3\sin 40°)$,

D $(C_x + 3\cos 60°,\ C_y + 3\sin 60°)$ and

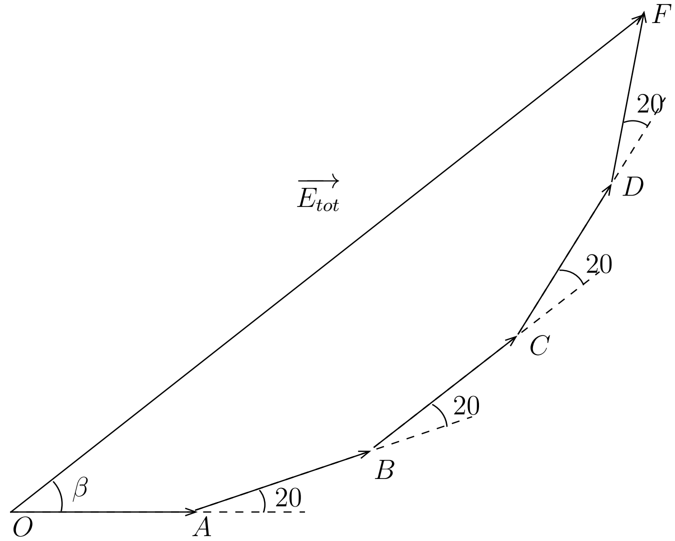
F $(D_x + 3\cos 80°,\ D_y + 3\sin 80°)$.

F (∼10.138, ∼ 8.506)

$OF = \sqrt{(F_x^2 + F_y^2)} \simeq 13.234$

$\tan\beta = F_y/F_x \Rightarrow\ \beta \simeq 40.0°\ \ (2\pi/9)$

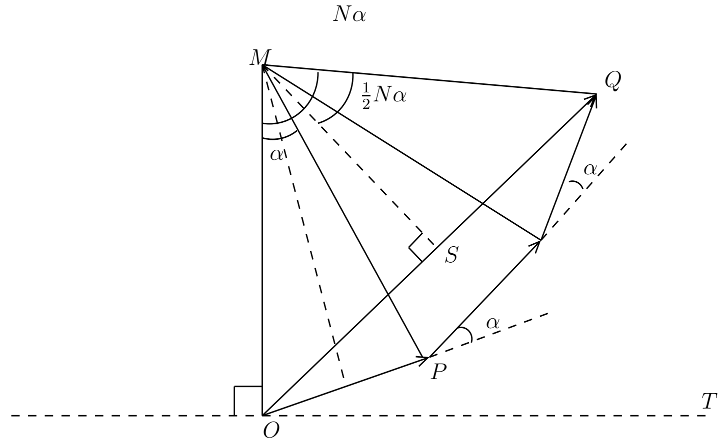
Thus $E(t) \simeq 13.234\cos(\omega t + 2\pi/9)$



b) Let $MO = R$. Adding the N vectors, we end at Q. All the tips lie on a circle with center at M. It follows from △ MOP: $OP/2 = R\sin(\alpha/2)$. It follows from △ MOQ: OQ/2=Rsin($N\alpha$/2).

Eliminate R: $OQ = OP\dfrac{\sin(\frac{1}{2}N\alpha)}{\sin\frac{1}{2}\alpha}$

OQ is ahead of OP by phase angle QOP. $\angle QOP = \angle QOT - \angle POT = (N\alpha - \alpha)/2$. To see that $\angle QOT = N\alpha/2$, draw the circle with center at M through O, P and Q; $OT \perp MO$ thus $\angle QOT = \angle OMQ/2 = N\alpha/2$. So Q is ahead by phase angle $\alpha/2(N-1)$. Since $|OP| = A$ in this problem, we find:



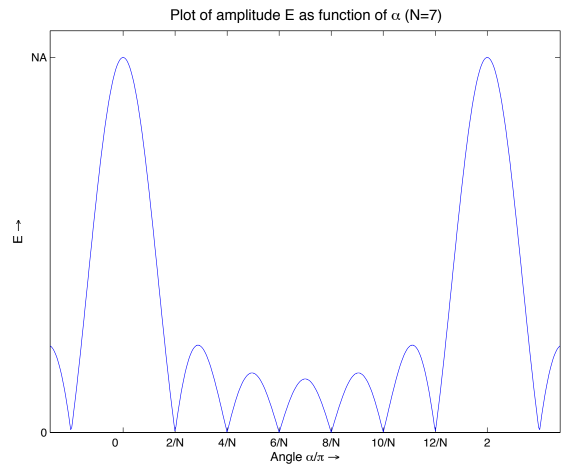
$$E(t) = A\frac{\sin(N\alpha/2)}{\sin(\alpha/2)}\cos\left[\omega t + \frac{1}{2}\alpha(N-1)\right]$$

Let us now test our result of Part (a): N=5 and $\alpha = \pi/9$. $N(\alpha - 1)/2 = 2\alpha$ (= 40°). The amplitude of the vector is (A=3) $3\sin(2.5\pi/9)/\sin(\pi/18) \simeq$ 13.234. On the button!

By adding vectors, point Q 'marches' on the circumference of the circle and will reach O (amplitude $E = 0$), then it traces its old route; a maximum is reached when Q is above M along the line OM. At that point, the amplitude ($2R$) is $OP/\sin(\alpha/2)$ and thus depends on α.

c) Let us now plot the vector amplitude as a function of α. When $\alpha = 0$ the vectors all line up (they are in phase) and we obtain the largest amplitude possible. This amplitude then should be N times the individual value of A; thus NA. Indeed, this can be found from the answer to Part (b). For α = 0, 2π, 4π, 6π etc the "upstairs" and "downstairs" of the amplitude of E are zero. Applying l'Hôpital's rule: $\lim\limits_{\beta\to\pi}\dfrac{\sin N\beta}{\sin\beta} = \lim\limits_{\beta\to\pi}\dfrac{N\cos\beta}{\cos\beta} = N$.

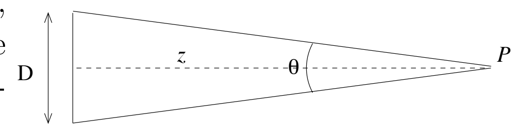
Thus, our function has a maximum amplitude of NA when $\alpha = 0,\ 2\pi,\ 4\pi, \ldots$. The amplitude is zero whenever $\sin(N\alpha/2) = 0$, so for $\alpha = 2\pi/N,\ 4\pi/N$ etc. *However, when at the same time* $\sin(\alpha/2) = 0$, *the amplitude is a maximum!* Thus, there are minima when $\alpha = 2n\pi/N$ (n is an integer) except when $n = N,\ 2N,\ 3N, \ldots$ n the figure, $N = 7$. Notice that there are $N - 1 = 6$ minima in between the main maxima. I have plotted $|\vec{E}|$. The light intensity is proportional to $|E|^2$ and thus is proportional to N^2 at maximum. By going from a radio interferometer with five dishes to one of ten dishes, the signal strength received increases by $10^2/5^2 = 4$ (assuming the dishes are all the same). I



Problem 9.5 — Think big

a) An EM plane wave of wavelength λ passes through a circular aperture of diameter D and diffracts. If we measure the intensity of the diffracted light at a distance z away from the aperture, such that $z \geq 2D^2/\lambda$, then we will observe Fraunhofer diffraction.

b) If we are located at P, a distance z from the aperture, $\sin\theta \approx D/z$ (see figure). For Fraunhofer diffraction, the first zero in intensity next to the prime maximum is observed when $\sin\theta \approx 1.2\lambda/D$. We therefore require that



$D/z \ll 1.2\lambda/D$. If this were not the case, the zero in intensity would be blurred due to the extended size of the aperture. Thus $z \gg D^2/1.2\lambda$. Perhaps somewhat arbitrarily, $z > 2D^2/\lambda$ is generally adopted.

c) An approximation of the minimum distance between the photographic plate and the slit is given by the Fraunhofer diffraction relation, $z_{\text{min}} \approx \dfrac{2D^2}{\lambda} \approx 5.8 \times 10^8$ m. This is farther than the distance to the Moon!

d) Our guess is that the central maximum will be about the same as the size of the aperture, since the Fraunhofer condition is just met; thus ≈ 12 m. A more formal calculation supports that. For small ϕ, $\sin\phi \approx \phi$. Using $\sin\phi = \lambda/D$, $\phi \approx \lambda/D \approx x/z$. Hence, $x = z\lambda/D$. Using $z = z_{\text{min}}$, the width of the central maximum is approximately 24 m.

e) We are now told that the new slit width is $D' = 2$ m. Hence, the width of the central maximum is now 6 times bigger, i.e. about 6×24 m $= 144$ m.

f) The aperture is now way too large to meet the Fraunhofer condition. Thus, the bright maximum will be about 96 m wide.

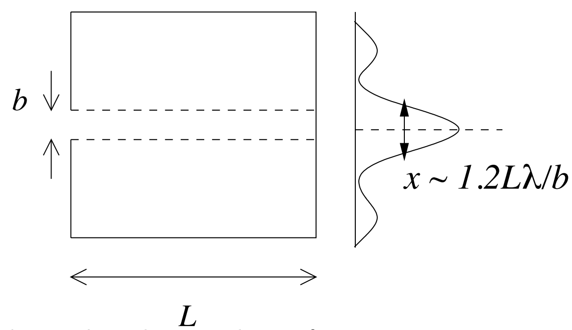
g) Alignment of the earth-star would be hopeless and contamination of your pattern by neighboring stars (after all, one does not simply aim a 12 m slit at a star) would make your task of differentiation impossible—not to mention problems of intensity.

Problem 9.6 (Bekefi & Barrett 8.7) — Angular resolution

In order to resolve the two light sources, we must be able to differentiate the two diffraction patterns on the objective of the telescope. Let's first see whether the 5 cm lens is capable of resolving the two lights. Its angular resolution is $\Delta\theta \approx \dfrac{1.2\lambda}{D} \approx 1.4 \times 10^{-5}$ rad.

The angular separation of 1 ft at a distance of 10 miles is about 1.9×10^{-5} rad. Thus, the telescope will be able to resolve the two sources of light. If we now place a slit in front of the lens, whose width is less than 5 cm, it will become more difficult to resolve the two lights. It's not clear now whether we should use as angular resolution λ/D or $1.2\lambda/D$. For a narrow long slit, we should use λ/D, but the length of the slit will NOT become much larger than its width, D. Therefore, we will stick (conservatively) with an angular resolution of $1.2\lambda/D$. Thus, we require $1.9 \times 10^{-5} > 1.2\lambda/D$. Hence, $D > 3.8$ cm.

Problem 9.7 (Bekefi & Barrett 8.8) — Pinhole camera

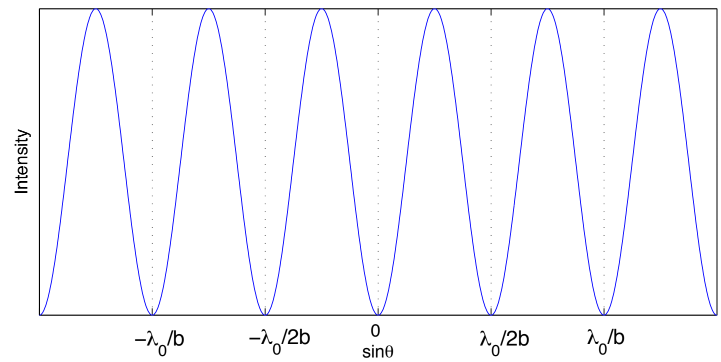


The figure shows the setup of the camera. The diffraction pattern is shown on the right. A distant source produces a Fraunhofer diffraction pattern on the screen with a central maximum of width $w \approx 1.2L\lambda/b$. This holds if, and only if, the coherence relation $\dfrac{b}{L} < \dfrac{\lambda}{b}$ is satisfied. Otherwise, the pattern will be washed out and we are dealing with Fresnel diffraction. The "blur" that appears on the screen will then have a width of about b. This is clear if you imagine moving the screen closer to the slit (imagine $b = 1$ cm). Then, we expect to see a light spot of width b on the screen. Hence, the diffraction pattern width given by the first equation does not hold. For example, let $\lambda = 500$ nm, $L = 1$ m and $b = 1$ cm. Then, the Fraunhofer diffraction width is $w \approx 60$ μm. However, in this case, the second equation does not apply. Notice that the coherence relation $L > b^2/\lambda$ dictates that $L > 10^{-4}/5 \times 10^{-7} \approx 200$ m, which is substantially larger than 1 m. The thought that you might see a spot with a width of about 60 μm is absurd! Instead, you will see a spot with a width of about 1 cm (Fresnel diffraction).

You can now see that, starting at very small values of b for given L, the diffraction pattern will have a width of about $1.2L\lambda/b$. For increasing values of b, the spot width will decrease (non-intuitive!). Then a point is reached, for increasing b, when the coherence relation is no longer satisfied and the spot size will have a width of about b and increases as b increases. Thus, the spot size as a function of b has a local minimum. At this minimum, you have approximately achieved the best resolution possible with a pinhole camera. The smallest spot width will appear when $\dfrac{1.2L\lambda}{b} \approx b$. Thus, $b = \sqrt{1.2\lambda L} \approx \sqrt{\lambda L}$. Using $\lambda = 500$ nm, $L = 1$ m, the optimal size of the hole $b \approx 0.8$ mm. Can you think of a way to do an experiment at home to demonstrate this phenomenon?

Problem 9.8 (Bekefi & Barrett 8.9) — Double slit interference

a) The dielectric slab in the slit effectively changes the optical path length, i.e. it adds an extra phase to the waves that pass through it. We can imagine an "air plate" of thickness d over slit A and a dielectric plate with the same thickness over slit B. When waves emerge from the plate at A, they have traveled a distance d which is equivalent to a phase angle $2\pi d/\lambda_0$.



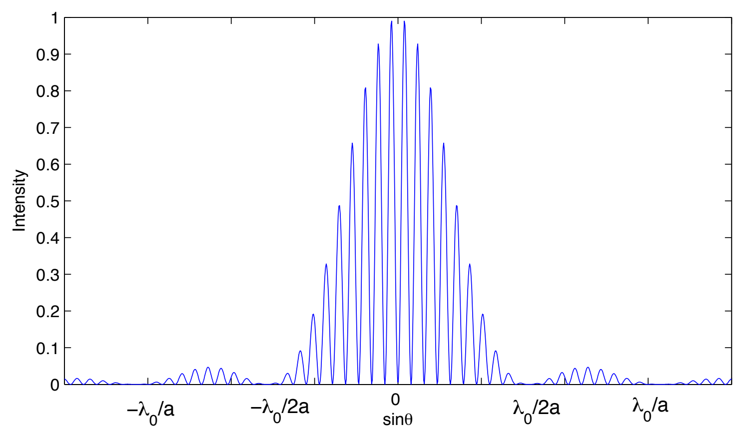
The waves that emerge from plate B have traveled a distance d which is equivalent to a phase difference of $2\pi nd/\lambda_0$. Here n is the index of refraction of the dielectric slab. We are given: $\frac{d}{\lambda_0}(\sqrt{\kappa}-1) = 5/2$. So, $d = \frac{5/2\lambda_0}{n-1}$, where the index of refraction $n = \sqrt{\kappa}$.

The diffraction pattern depends on the phase difference δ between the waves emerging from the two slits. In this case, $\delta = 2\pi\frac{d}{\lambda_0}(n-1)$, which means $\delta = 5\pi$ for the specific values of d and κ.

Thus, the Huygens sources at the 2 slits after traveling the distance d are out of phase by π. Hence, the diffraction pattern has been shifted by π. So, there will be a minimum at $\theta = 0$ and there will be maxima at angles θ_m such that $2b\sin\theta_m = (2m+1)\lambda_0/2$. The plot shows intensity vs $\sin\theta$.

b) Diffraction causes the interference pattern to be modulated with a term $\sin^2(\beta)/\beta^2$, where $\beta = (2\pi a/\lambda_0)\sin\theta$. Hence, considering both interference and diffraction, the intensity pattern is:

$$I = 4I_0\left(\frac{\sin(2\pi a\sin\theta/\lambda_0)}{2\pi a\sin\theta/\lambda_0}\right)^2\cos^2\left(\frac{2\pi b}{\lambda_0}\sin\theta - \frac{5}{2}\pi\right),$$



where I_0 is the maximum intensity (W/m^2), i.e. the intensity of light if there were only one slit. Note the $5\pi/2$ phase shift in the cosine term due to the dielectric slab. The modulation due to the slit width produces a first minimum when $(2\pi a/\lambda_0)\sin\theta = \pi$, or $\sin\theta = \lambda_0/2a$.

Since $b/a = 10$, $\sin\theta \approx \theta \approx 10\lambda_0/2b = 5\lambda_0/b$. A plot of intensity vs $\sin\theta$ is shown.