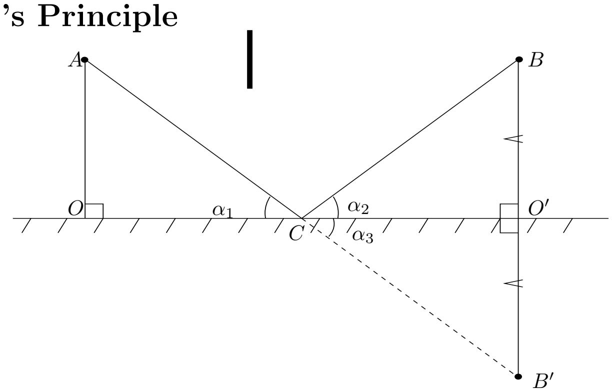
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #8 Solutions

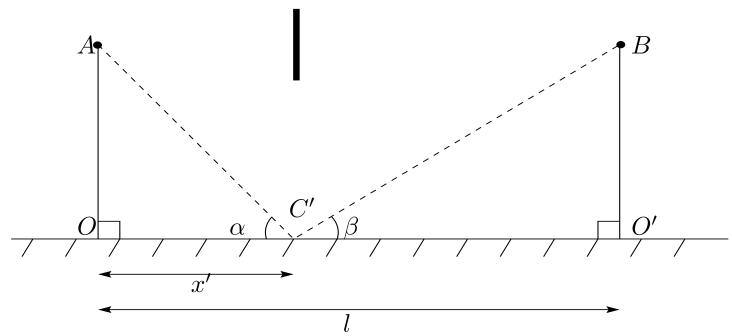
Problem 8.1: Nature is in a hurry - Fermat's Principle

a) See figure.



b) $BO' = B'O'$ and angle $BO'C=$ angle $B'O'C$. So $\triangle BO'C$ is congruent to $\triangle B'O'C$.
Hence $\alpha_2 = \alpha_3$ (Congruency), $\alpha_1 = \alpha_3$ (Opposite angles), $\Rightarrow \quad \alpha_1 = \alpha_2$. The angle of incidence is the same as the angle of reflection.

c) The figure shows the setup when the hypothetical light ray reflects off at a random point C'. The distance S the light ray has to travel from A to B is:



$$\begin{aligned} S &= AC' + C'B \\ &= \sqrt{d^2 + x'^2} + \sqrt{d^2 + (l - x')^2} \\ \frac{dS}{dx'} &= 0 \text{ for minimum time of travel} \end{aligned}$$

$$\begin{aligned} \frac{dS}{dx'} &= \frac{1}{2}\frac{1}{\sqrt{d^2 + x'^2}}(2x') + \frac{1}{2}\frac{1}{\sqrt{d^2 + (l - x')^2}}(2(l - x'))(-1) = 0 \\ \Rightarrow &\quad \frac{x'}{\sqrt{d^2 + x'^2}} - \frac{(l - x')}{\sqrt{d^2 + (l - x')^2}} = 0 \end{aligned}$$

which means that $\cos\alpha - \cos\beta = 0$ so $\cos\alpha = \cos\beta$ or $\alpha = \beta \Rightarrow \; x' = l/2$.

d) If A is a distance d above the horizontal and B' is a distance d below the horizontal, the line AB' will intersect the mirror at midpoint C of OO' such that $x_c = l/2$, same as for C' from Part(c).

Problem 8.2: (Bekefi & Barrett 7.4)[1] Fiber optics

See figure on the next page. To ensure that all light entering at A emerges at B, it is sufficient that the smallest incidence angle θ of the incoming beam be greater than the critical angle θ_c for the medium. The first bounce has the smallest angle of incidence. It is larger for the following bounces. Therefore, the light will arrive at the end of the light pipe, when the angle of incidence for the first bounce is larger than the critical angle for total internal reflection.

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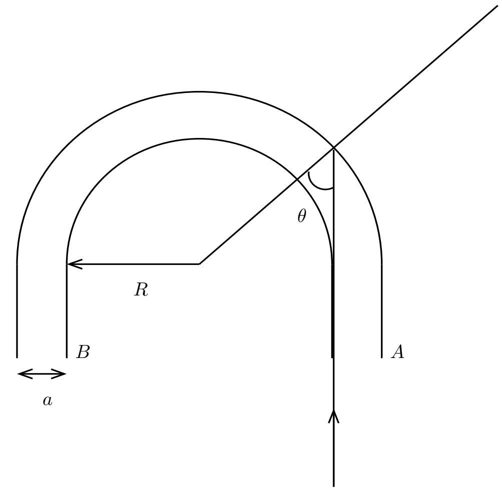
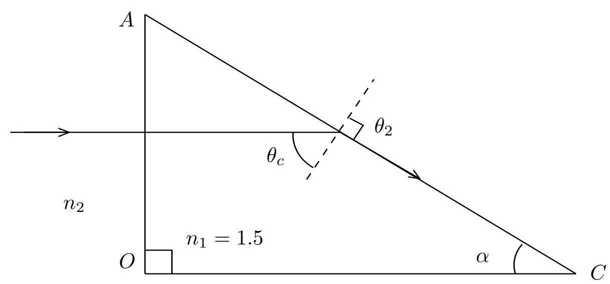
[1] The notation "Bekefi & Barrett" indicates where this problem is located in one of the textbooks used in 8.03 in 2004: Bekefi, George, and Alan H. Barrett *Electromagnetic Vibrations, Waves, and Radiation.* Cambridge, MA: MIT Press, 1977. ISBN: 9780262520478.

This minimum incidence angle occurs when the incoming ray is grazing the inner curved surface as shown. Since all beams which are incident at A have incidence angles which are greater than the incidence angle of this ray, they will all be totally internally reflected if this ray is totally internally reflected. That is true for the following condition:
$\sin\theta \geq \sin\theta_c$ where $\sin\theta = \dfrac{R}{R+a}$ and $\sin\theta_c = \dfrac{1}{\eta}$.
So, $\dfrac{R}{R+a} \geq \dfrac{1}{\eta}$ and therefore $a \leq (\eta - 1)R$ where $\eta = 1.5$
and so $a_{max} = \dfrac{R}{2}$



Problem 8.3: (Bekefi & Barrett 7.5) Total reflection

a) The index of refraction of the prism material is $n_1 = 1.5$. For total reflection at side AC, the critical incidence angle θ_c is such that $\theta_2 = 90°$. $n_1 \sin\theta_c = n_2 \sin\theta_2$ where $\theta_2 = 90°$ so $n_1 \sin\theta_c = 1$ and $\sin\theta_c = 1/n_1$ Since $\theta = \pi/2 - \alpha$, we know that $\cos\alpha_c = 1/n_1$ where $n_1 = 1.5$ and so $\alpha_c = 48.2°$



b) For the light to be totally internally reflected, $\theta > \theta_c \;\Rightarrow\; \alpha < \alpha_c$. It is a maxima and the light can undergo total internal reflection only for prism angles $\alpha < \alpha_c = 48.2°$.

Problem 8.4: Light under water

a) Red light ($\lambda \approx 650$ nm) is incident on the water surface of a swimming pool. The frequency of the light is unchanged after refraction at a medium interface. Hence: $\nu = \nu'$ which means $\dfrac{v}{\lambda} = \dfrac{v'}{\lambda'}$ where $v = \dfrac{c}{n}$. This means that $n\lambda = n'\lambda'$ where $n = 1$ and $n' = 1.33$. So, using $\lambda \approx 650$ nm, $\lambda' = \dfrac{n}{n'}\lambda \simeq 488$ nm. If we swim under water and look up at the refracted light coming from the surface, we will still see red light. One way of looking at this is that the frequency remains the same, thus our brains process the signal the same way. Another way of thinking is to calculate the wavelength as the radiation reaches our retina. Convince yourself that this is the same whether you are under water or above water.

b) The Fresnel equations reduce to the following reflection and transmission coefficients for normal incidence ($\theta_{incidence} = 0$) : $r_\perp = \dfrac{n_1 - n_2}{n_1 + n_2} \quad t_\perp = \dfrac{2n_1}{n_1 + n_2}$
Now the power of electromagnetic waves in a medium of refractive index n is given by $n|E|^2/c$.

Let the power in the incident, reflected and transmitted light be P_i, P_r and P_t respectively. Then the sum of the total power in the reflected and the transmitted light is:

$$\begin{aligned}
P_r + P_t &= \frac{n_r}{c}|E_r|^2 + \frac{n_t}{c}|E_t|^2 = \frac{n_1}{c}{r_\perp}^2|E_i|^2 + \frac{n_2}{c}t_\perp^2|E_i|^2 \qquad\qquad n_r = n_1 \quad n_t = n_2 \\
&= \frac{|E_i|^2}{c}\left[n_1\left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 + n_2\left(\frac{2n_1}{n_1 + n_2}\right)^2\right] \\
&= \frac{|E_i|^2}{c}\left[\frac{n_1(n_1^2 + n_2^2 - 2n_2n_1 + 4n_2n_1)}{(n_1 + n_2)^2}\right] = \frac{|E_i|^2}{c}\left[\frac{n_1(n_1 + n_2)^2}{(n_1 + n_2)^2}\right] = \frac{n_1}{c}|E_i|^2 \\
\Rightarrow \qquad P_r + P_t &= P_i
\end{aligned}$$

As expected, total energy is conserved.

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