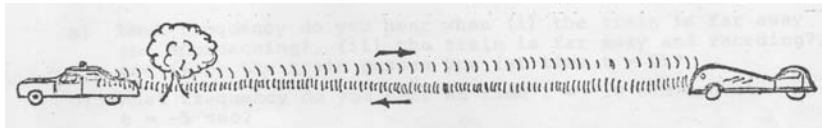
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #7

Problem 7.1 – Speed checked by radar

A radar beam ($\lambda = 3$ cm) is reflected off a car as shown in the figure (head-on reflection).



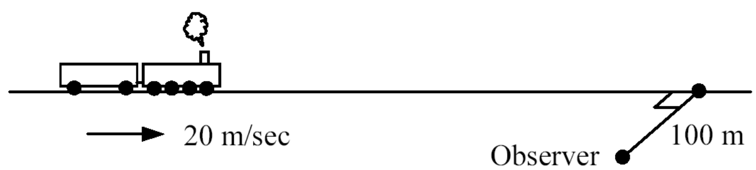
The radar transmitter is not moving. The reflected beam is again received by the police; the frequency f' of this reflected signal is measured by a stationary receiver on the police car. From this, the velocity v of the car is calculated. All this is done in a "black-box" in less than a small fraction of a second! A digital display tells the police officer your speed nicely converted into miles/hour.

a) What is the frequency of this radar transmitter?

b) Give an equation that relates f, f' and v. (*Hint: first calculate the frequency that the car "sees", then send this signal back to the police.*)

Problem 7.2– Can't you hear the whistle blowing?

A train travels down a long straight track at a speed of 20 m/sec (∼45 mph) blowing its whistle constantly with a frequency of 1000 Hz. You, the observer, are standing at a point 100 m from the track (see figure). Define time $t = 0$ as the moment that the train is closest to you.



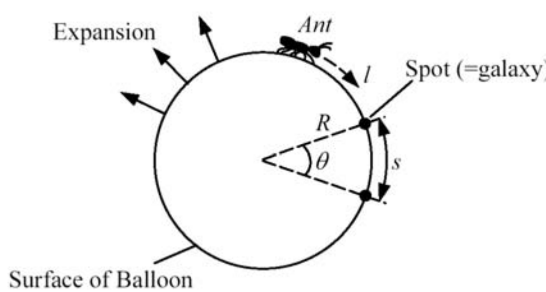
a) What frequency do you hear when (i) the train is far away and approaching?, (ii) the train is far away and receding?, and (iii) the train passes you (at time $t = 0$)?

b) What frequency do you hear at time $t = -10$ sec and at $t = -5$ sec?

c) Sketch the curve of the frequency that you hear versus time.

Problem 7.3 – Our expanding universe – simplified

The figure illustrates the famous expanding balloon analogy for our universe. All of space is represented by the surface of the spherical balloon, and clusters of galaxies are represented by spots painted on this surface; the radius of the balloon corresponds to $R(t)$, the "radius" of the universe. As the balloon expands, the spots remain at constant angular separations (θ) from one another. Let the balloon expand at a constant rate.



a) Prove that $\left(\dfrac{\Delta s}{\Delta t}\right) = \left(\dfrac{1}{R}\dfrac{\Delta R}{\Delta t}\right)s$ where s is the separation between any two spots (on the surface) and $(\Delta s/\Delta t)$ is the speed of recession of one spot from another. Note that this equation is equivalent to Hubble's Law: $v = Hr$

b) What is H in terms of the symbols used in the first equation? Check that your answer is dimensionally correct; H in general is expressed in km/sec per Mpc.

The light photons from distant galaxies may be represented by "ants" crawling along the balloon's surface at speed l.

c) Show that for uniform cosmic expansion [i.e., $(\Delta R/\Delta t) =$ constant] there is a distance (s) from beyond which these ants cannot ever reach our galaxy (this distance is called the horizon)!

Let us now turn to the way we looked at our world a few years ago, before the discovery of "dark energy". Dark energy is responsible for the fact that the expansion of the universe, according to present belief, is accelerating, not decelerating. This mysterious dark energy is presently one of the "hottest" topics in all of physics. In what follows, pretend that there is no dark energy.

Consider an expanding (gravitating) gaseous sphere of uniform mass density ρ, with a total mass M and a radius $R(t)$. A gas particle at the surface of this sphere will move radially outward with velocity v so that $\dfrac{v^2}{2} = \dfrac{GM}{R} + constant$ where $v = \Delta R/\Delta t$ is the radial speed. This must be a familiar equation from your 8.01 (Newtonian mechanics) days. If the constant is positive then the expansion will never stop (too much kinetic energy); if the constant is negative then the expansion will ultimately stop and a collapse under the influence of gravitational forces will follow.

d) Show that the equation above can be written in the form: $\left(\dfrac{1}{R}\dfrac{\Delta R}{\Delta t}\right)^2 = \dfrac{8\pi G\rho}{3} + \dfrac{2(constant)}{R^2}$

In the following we will introduce a subscript 0 when we refer to our present time.
Assume $H_0 = 70\ \frac{\text{km/sec}}{\text{Mpc}}$ and $G = 6.7 \times 10^{-11}$ (SI units).

e) What is the minimum required density of our universe now (ρ_0) to make it closed?

f) In the equation in part (d), replace ρ by $\dfrac{M}{\frac{4}{3}\pi R^3}$, where M is the total mass of the universe. Show that in case of a flat universe the radius of the universe $R(t) \propto t^{2/3}$. From this it follows immediately that $H = \dfrac{2}{3t}$ (show that) and consequently the age of our universe is now $t_0 = \dfrac{2}{3H_0} \sim 9.3 \times 10^9$ years.

g) Your modified equation relates $H^2 = \left(\dfrac{1}{R}\dfrac{\Delta R}{\Delta t}\right)^2$ with R. Show that Hubble's constant must have been larger in the past.

h) Could H become negative? What consequence would that have for our "red-shifted" galaxies.

Problem 7.4 – Doppler shifts of EM radiation $\Rightarrow$ a black-hole X-ray binary

One star in an X-ray binary system (the donor, with mass m_1) is only detected in the optical band. The other (the accretor, with mass m_2) is only detected in X-rays. The orbits are circular, the radii are r_1 and r_2, respectively. The optical observers conclude from a close inspection of the optical spectrum that m_1 is approximately 30 times more massive than our sun (it is a super giant).

a) Derive the orbital period T in terms of $m_1,\ m_2,\ r_1,\ r_2$, and G.

A particular absorption line in the visible spectrum moves back and forth periodically (in a sinusoidal fashion) with a period of 5.6 days. The minimum and maximum observed wavelengths of the line are 499.75 nm and 500.25 nm, respectively. Assume that we observe the binary edge on.

b) What is the speed of the donor in its circular orbit?

c) Calculate r_1.

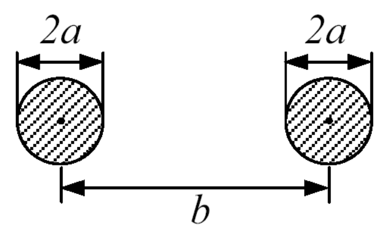
d) Calculate r_2. Your calculations will be greatly simplified if you set up your equations in terms of r_2/r_1. You will find a third order equation in r_2/r_1. Only one solution is real. There are various ways to find a decent approximation for r_2/r_1: (i) trial and error using your calculator, (ii) plot the function, (iii) MatLab.

e) Calculate the mass m_2 of the accretor.

Since the accretor must be compact (we observe a strong flux of X-rays) and because its mass is substantially larger than 3 times the mass of the sun (this is the maximum mass for a neutron star), it is very likely that the accretor is a black hole. A result somewhat similar to this simplified example was first published in 1972 by Bolton and independently by Webster and Murdin for the X-ray binary system Cyg X-1.

Problem 7.5 (Bekefi & Barrett 5.3)[1] – Transmission line

A transmission line consists of two parallel wires each of radius a. The distance between the centers of the wires is b.



a) Assuming that $b \gg a$, show that the capacity and inductance per unit length of the line are approximately:

$$C_0 \simeq \frac{\pi\epsilon_0}{\ln(b/a)} \quad \text{and} \quad L_0 \simeq \frac{\mu_0}{\pi}\ln(b/a) \quad \text{respectively.}$$

Notice that the units of C_0 are Farad/m (same as ϵ_0). The units of L_0 are Henry/m (same as μ_0)

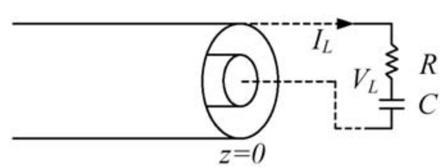
b) Using the results of part(a), compute the phase velocity v of a wave propagating on the line.

c) Obtain an expression for the characteristic impedance Z_0.

d) The parallel wire transmission line is made from No. 12 wires (diameter 0.0808 inches) spaced 0.50 inches apart. Calculate $C_0,\ L_0,\ v$ and Z_0.

Problem 7.6 – Coaxial cable

A coaxial cable with characteristic impedance Z_0 is terminated by a series combination of a resistor and a capacitor. If a harmonic voltage wave is incident from the left, a reflected wave will be set up by the load. The resulting total voltage on the



[1]The notation "Bekefi & Barrett" indicates where this problem is located in one of the textbooks used in 8.03 in 2004: Bekefi, George, and Alan H. Barrett *Electromagnetic Vibrations, Waves, and Radiation*. Cambridge, MA: MIT Press, 1977. ISBN: 9780262520478.

line will have the form $V(z,t) = V_i e^{j(\omega t - kz)} + V_r e^{j(\omega t + kz)}$

a) Write down an expression for the current $I(z,t)$ on the line.

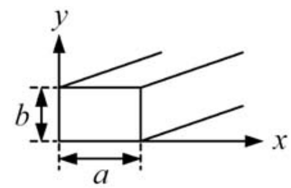
b) Find the relation between the complex voltage across the load, V_L, and the complex current into it, I_L.

c) Find V_r in terms of V_i, ω, R, C, and Z_0 using the boundary conditions on voltage and current. *Comment: Notice that it is complex, indicating that the load can change both the amplitude and the phase of the reflected wave.*

d) Is your result in (c) consistent with the general relationship $\dfrac{V_r}{V_i} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$?

e) Sketch the amplitude and the phase of the reflected voltage wave as a function of frequency ω for the special case $R = Z_0$.

Problem 7.7 (Bekefi & Barrett 5.4) – Rectangular waveguide



A waveguide of rectangular cross section operates in the TE_{mn} mode with $E_y = E_{0y}\sin(k_x x)\cos(k_y y)\cos(\omega t - k_z z)$. The field distribution must satisfy the wave equation and boundary conditions at the faces of the guide tube.

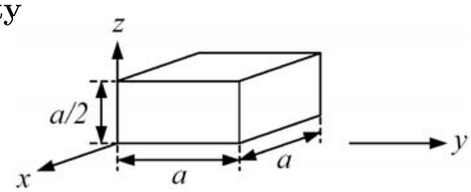
a) Using the wave equation, develop the necessary relationship between the frequency ω and the various wave numbers.

b) Using boundary conditions at the faces $x = 0$ and $x = a$, show what restrictions on the wave numbers are required.

c) Using boundary conditions at the faces $y = 0$ and $y = b$, show what restrictions on the wave numbers are required.

d) Show that there is a minimum frequency for which propagation will occur and determine this for the TE_{mn} mode.

Problem 7.8 (Bekefi & Barrett 5.7) – Resonance cavity



A copper box with dimensions as shown in the figure acts as a cavity resonator. $E_z = E_0 \sin(k_x x)\sin(k_y y)\sin(\omega t)$, and $E_x = E_y = 0$ is one solution of the wave equation for this case.

a) Find the lowest resonance frequency ω_1 and the corresponding free space wavelength λ_1.

b) Find the next-to-lowest resonance frequency ω_2 and the corresponding wavelength λ_2.

Problem 7.9 – Radiation pressure

A perfectly reflecting mirror of mass $M = 1$ g hangs vertically from a wire of length $L = 10$ cm. It is illuminated with a constant laser beam of intensity 30 kW (a powerful laser!), incident normal to the surface of the mirror. What is the displacement of the mirror from its equilibrium position?

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