Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #17: Wave Guides & Resonant Cavities

The discussion of electromagnetic wave propagation can be expanded from the previous one-dimensional analysis to higher dimensions. Normal modes were already discussed in higher dimensions but not in the context of waves. Much as in that case, the transition from one dimension to three dimensions is straightforward. The scalar wavenumber becomes a wave vector, represented in terms of its components along $(\hat{x}, \hat{y}, \hat{z})$. The wave vector is in the direction of propagation of the wave in three dimensions $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ (**2:00**). The wavelength, λ, remains $\lambda = 2\pi/k$, with k the magnitude of the wave vector (still called the wavenumber), i.e. $k = |\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The speed of propagation for light in vacuum (nondispersive) is c and the phase velocity equation $v = \omega/k$ gives $\omega = kc$.

Consider EM waves polarized in the y direction and propagating in the z direction between two plates which are very large in these two directions and located at $x = 0$ and $x = a$. ("large" usually means with respect to the wavelength.) Recall the boundary condition that at the surface of a conductor the parallel component of the electric field must be zero, so $E_y = 0$ at both plates. The k vector cannot have a y component but could have a component in the x direction (**5:00**), so one can regard the wave as bouncing between the plates much like a beam of light between two parallel mirrors. Planes of constant phase are perpendicular to $\vec{k}$. By drawing out the wave fronts, it is clear that when an advance of one wavelength is made along the direction of $\vec{k}$, a larger distance L_z is moved along the z direction. The wave phase advances in the z direction in the ratio L_z/λ faster than in the actual direction of propagation. An analogy was made previously to waves coming into a shore where the edge of a breaking wave can move very fast parallel to the shoreline compared to the speed with which the wave crests are moving. Taking the direct ratio of L_z/λ, or the inverse ratio of k's, the phase velocity in the z direction is larger than the speed of light: $v_{pz} = \frac{L_z}{\lambda}c = \frac{k}{k_z}c > c$. Conversely, the rate of progress of the overall wave, zigzagging down the slot (the group velocity) is slower: $v_{gz} = (k_z/k)c < c$ (**10:00**).

Recall that the wave equation for the electric field is $\nabla^2\vec{E} = \dfrac{\partial^2 E_y}{\partial x^2} + \dfrac{\partial^2 E_y}{\partial z^2} = \epsilon_0\mu_0\dfrac{\partial^2 E_y}{\partial t^2}$. The solution in this particular case will be a standing wave in the x direction and a traveling wave in the z direction, $\vec{E}(x, z, t) = E_{0y}\cos(\omega t - k_z z)\sin(k_x x)\hat{y}$. The dispersion relation is $\omega^2 = (k_x^2 + k_z^2)c^2$ and $c^2 = 1/(\epsilon_0\mu_0)$. The boundary conditions at the plates require that $k_x = n\pi/a$ where n is 1, 2, 3... (**14:00**) The electric field amplitude has the shape of a standing wave across the slot. The resulting k vector is $\vec{k} = (n\pi/a)\hat{x} + k_z\hat{z}$ and so: $\omega = kc = c\sqrt{\left(\dfrac{n\pi}{a}\right)^2 + k_z^2}$. (**16:50**)

The phase velocity along z was deduced previously from geometry, $v_{pz} = \omega/k_z = kc/k_z > c$, and the group velocity, $d\omega/dk$, is also that found from the geometry, i.e. $v_{gz} = (k_z/k)c < c$. The lowest frequency possible is $\omega_c = c\pi/a$ corresponding to a wavelength $\lambda = 2a$. Propagation can occur at any frequency above this cutoff (but with $n = 1$ corresponding to a half-wave standing wave in x) (**20:15**) . The dispersion curve between k_z and ω (with $k_x = \pi/a$ for the half wave) is such that ω/k (the phase velocity) is always greater than c while $d\omega/dk$ (group velocity) is always smaller than c. A description (referred to as a *gedanken* or thought experiment) of the behavior of the system as the frequency is decreased from some large value follows (**24:00**). The cases of $n = 2...$ give dispersion curves at higher frequencies (**27:30**). It is possible for a single frequency to occur for two different modes with the same k but different k_x, i.e. different standing wavelength in x.

If the polarization direction is changed, the boundary conditions, which depend on the normal component, no longer restrict things and any frequency can propagate. This effect could even act to separate polarization components (**33:00**). A demonstration is done using a fixed frequency of 10^{10} Hz ($\lambda \sim 3$ cm) but varying the size of the gap to vary the cutoff point. Propagation ceases at the expected gap size of 1.5 cm. With polarization perpendicular to the plates, the size of the gap determines the intensity but there is no cutoff (**37:00**).

Now consider a closed box with conducting walls so that E_x is 0 at y=0 and b and also at $z = 0$ and c. We can immediately write $E_x = E_{0x} \sin(k_y y) \sin(k_z z) \cos(\omega t)$ with $k_y = m\pi/b$, $k_z = n\pi/c$ where $m = 1, 2, 3...$ and $n = 1, 2, 3...$, $\vec{k} = k_y \hat{y} + k_z \hat{z}$ (**43:20**). This can be shown to be a solution to the wave equation and the dispersion relation is simply $\omega^2 = (k_y^2 + k_z^2)c^2 = k^2c^2$ or $\omega_{m,n}^2 = \left[(m\pi/b)^2 + (n\pi/c)^2\right] c^2$. It is unfortunate that one of the lengths of the box was labeled "c" like the speed of light but this should not be a problem in context (**47:00**). To meet the boundary conditions, the frequencies are quantized. In three dimensions, there are no longer nodal lines but rather nodal surfaces but other aspects are similar to vibrations of membranes (with nodal lines) or even to strings (nodes or nodal points).

Similarly, $E_y = 0$ at $x = 0$ and a and also at $z = 0$ and c. The solutions are very similar to those for E_x, and we find $\omega_{l,n} = \left[(l\pi/a)^2 + (n\pi/c)^2\right] c^2$, another infinite family of values (**51:30**). The general solution for linearly polarized radiation in any direction (including also a z component) has the dispersion relation $\omega_{l,m,n} = \left[(l\pi/a)^2 + (m\pi/b)^2 + (n\pi/c)^2\right] c^2$.

Sound, which is a longitudinal wave with alternating small amounts of excess and diminished air pressure, has different boundary conditions (**54:00**). Also, polarization does not come into play. Particles at the walls cannot move through the walls so there can be no flow there, but the wall pressure can change in response to sound waves. Thus we should have maximum pressure changes, i.e. antinodes at the walls. This boundary condition, antinodes instead of nodes at the walls, changes the sine functions in the previous solution to cosines: $p = p_0 \cos(k_x x)\cos(k_y y)\cos(k_z z)\cos(\omega t)$.

For antinodes at the boundaries, $k_x = l\pi/a$, $k_y = m\pi/b$, and $k_x = n\pi/c$. The dispersion relation is $\omega^2 = k^2 v^2$ with v the speed of sound, and as usual $\lambda = 2\pi/k$. The full dispersion relation $\omega^2_{l,m,n} = \left[(l\pi/a)^2 + (m\pi/b)^2 + (n\pi/c)^2\right] v^2$. The usual unit for sound frequencies is Hz (**57:45**):

$$f_{l,m,n} = \frac{\omega_{l,m,n}}{2\pi} = \frac{v}{2\pi}\sqrt{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} = \frac{v}{2}\sqrt{\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{n}{c}\right)^2}$$

Based on this, the resonance frequencies of sound in a square box can be predicted. A demonstration is done with a box of $30.0 \times 39.85 \times 50.0$ cm, all uncertain to 0.1 cm. These are the lengths a, b, and c which correspond to the indices l, m, n, respectively in the table below. The speed of sound, $v_s \approx 344$ m/s, is proportional to the square root of the temperature (in K) and, therefore, accurate to about 1%. From this the predicted results can be tabulated in order of increasing frequency as the indices are varied, and then compared to what is actually observed (**1:02:00**).

$l\ m\ n$	Hz predicted	Hz observed
0 0 1	344	344
0 1 0	432	434
0 1 1	552	548
1 0 0	573	575
1 0 1	669	670
0 0 2	688	691
1 1 0	718	721
1 1 1	796	792
0 1 2	812	812

All observed resonant frequencies are within 1% of the prediction. A plot is then presented (**1:08:00**) showing the resonance spectrum as amplitudes as a function of frequency. Predicting the expected amplitudes is actually pretty hard.

The final demo shows the response to a transient 1.5 s pulse of 581 Hz injected into the box (**1:11:00**). This interferes with the normal mode solution, likely the 575 Hz resonance, to cause beats. The transient oscillation is 6 Hz as predicted by the 581-575 Hz frequency difference. When the driver is turned off, the system oscillates at 27 Hz, which seems to be the 575 Hz normal mode beating with the 548 Hz normal mode.

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