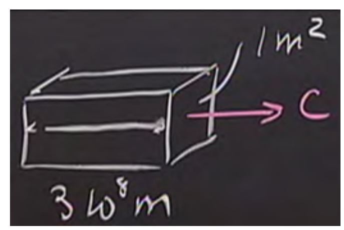
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #14: Generating EM Waves, Energy, Scattering

There is a local energy density wherever there is an electric or magnetic field. The energy density, in SI units, is measured in joules per cubic meter, i.e. $\mathrm{Jm^{-3}}$. For an electric field, it is given by $u_E = \frac{1}{2}\epsilon_0 E^2$ and since "there is no such thing as a free lunch", reflects the work done to bring charges together that create the field (**1:00**). Similarly, in making currents that create a magnetic field, work must be done, and that is reflected in the energy density in a magnetic field being $u_M = \frac{B^2}{2\mu_0}$. In a traveling EM wave, the magnitude of the magnetic field is related to that of the electric field by $|\vec{B}| = \frac{|\vec{E}|}{c}$ so, in that specific case, the magnetic energy density can be rewritten as $u_M = \frac{B^2}{2\mu_0} = \frac{E^2}{2\mu_0 c^2} = \frac{1}{2}\epsilon_0 E^2$ where to get the last form we used the fact that $c^2 = \frac{1}{\epsilon_0\mu_0}$. We see that the energy density in the electric field of a traveling wave is exactly the same as the energy density in its magnetic field. This reflects the wonderful symmetry that these fields are necessarily intertwined in the traveling wave solution of Maxwell's equations. The total energy density is thus (in vacuum) (**3:00**): $u_{tot} = \epsilon_0 E^2 = \epsilon_0 EBc$.



The wave, of course, moves and in so doing it carries energy with it. We would like to calculate how much energy flows through one square meter if an EM wave flows perpendicularly through it as shown. The dimensions of the expected result are $\mathrm{Js^{-1}m^{-2}} = \mathrm{Wm^{-2}}$. This rate of energy flow through one square meter can be expressed as the energy that was in the box and that flowed out through the end in one second: $u_{tot}c = \epsilon_0 EBc^2 = \frac{EB}{\mu_0}$. This is reminiscent of the Poynting vector $\vec{S}$ seen in earlier studies, where $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$ in $\mathrm{Wm^{-2}}$ (**5:00**).

Both $\vec{E}$ and $\vec{B}$ are time variable, so the Poynting vector also varies in time. Although the fields vary as $E = E_0\cos\omega t$ and $B = B_0\cos\omega t$, this rapid variation of the Poynting vector would only show up on spatial scales of less than a wavelength. The overall variation of the Poynting vector is the square of a cosine, and $\langle\cos^2\omega t = 1/2\rangle$. Taking this into account, the averaged magnitude of the Poynting vector is $\langle S\rangle = \frac{1}{2}\frac{E_0 B_0}{\mu_0} = \frac{E_0^2}{2\mu_0 c}$ in terms of the amplitudes of the fields, and where the latter form shows that it is only necessary to know the peak electric field amplitude in order to completely specify the Poynting vector. Once more, this reflects the necessary coupling of $\vec{E}$ to $\vec{B}$ and in fact a specification of B_0 would also be sufficient to specify the Poynting vector. If we consider a traveling EM wave in which $E_0 = 100$ V/m, then we can plug in values to get that (**8:00**): $\langle S \rangle = \frac{1}{2}\frac{100^2}{\mu_0 c} = 13$ Wm^{-2}.

If you expose your body to this energy flow of visible and infrared light, your body will absorb it (unlike X-rays or gamma radiation which would go right through you). This amount of this type of radiation is not harmful: the body itself emits about 100 W over an area of roughly one square meter, so this is small in comparison. If we now consider an electric field of $E_0 = 10^3$ V/m, the Poynting vector, which goes as the square of the electric field, becomes $\langle S \rangle = 1.3$ kW/m^2, which is a a dangerous amount of visible and infrared, potentially causing skin cancer or worse.

The Sun is a powerful light source, producing 3.9×10^{26} W, and Earth is 150×10^6 km away. We can calculate how many joules per second go through one square meter at this distance. The portion of the total emission that would go through 1 m^2 is the ratio of 1 m^2 to the total area of a spherical surface at the distance of the Earth from the Sun. We thus get that $S = \frac{3.9 \times 10^{26}}{4\pi(150 \times 10^9)^2} =$ 1.4 kW/m^2. That is why the number 1.3 kW/m^2 can be dangerous (**11:20**).

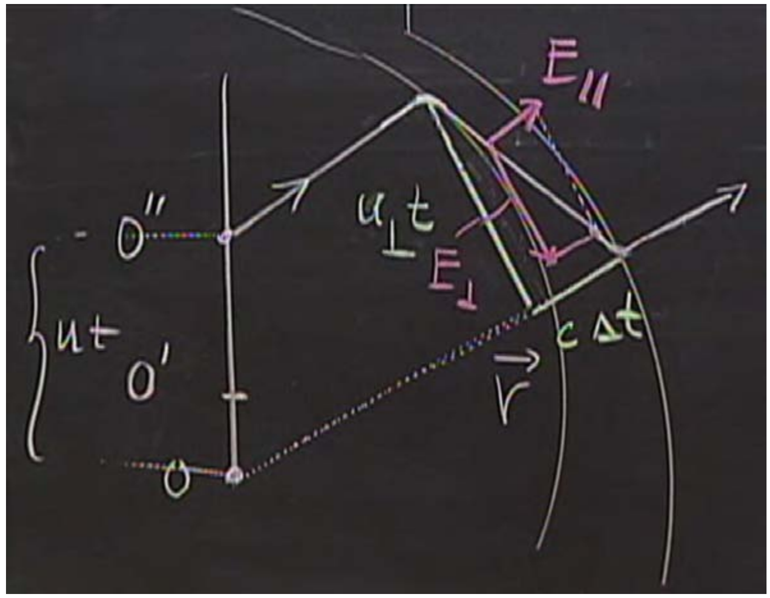
This number, called the solar constant, is very important for discussions of harvesting solar energy. For every square meter one can expose to solar energy, one can never get more than about 1400 joules per second of energy. The (2004) electric power capacity of the United States was about 700,000 MW, given by about 700 power plants, each of about 1,000 MW. The U.S. is energy hungry, consuming about one quarter of the Earth's electric power consumption. Basically, to get a lot of electricity from solar power, one would need hundreds of square km of very expensive solar cells in the desert. The efficiency is not 100% and the situation gets worse when the Sun is low in the sky. So, the solar constant imposes a limit on how much one can get from solar power (**13:00**).

This leads us to ask whether there really is such a thing as an electric field of 1000 V/m in the solar radiation. The radiation is not in the form of idealized plane waves, and there is not just one wave from the Sun having this large electric field amplitude. The Poynting vector from the Sun is 1.4 kW/m^2, but one cannot naively associate with that an EM field of 1000 V/m.

This brings us to the topic of how EM waves are produced. In a nutshell, this is due to the acceleration of electric charges. Charges that are stationary or moving at constant velocity are surrounded by radial electric fields. Whether at rest or in constant motion, there are radial field lines with no kinks in them (**15:00**). The moment a charge is accelerated, as is looked at in more detail below, there is a kink put into the field line, and that manifests itself as EM radiation.

We consider a charge q at the point O and initially at rest. It is now accelerated in the upward direction, with an acceleration $\vec{a}$ for Δt seconds. It ends up at location O'. The velocity at O'

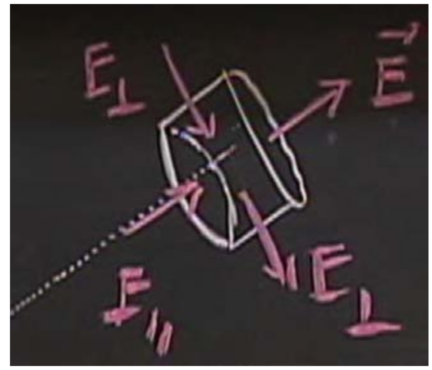
will be upward, $\vec{u} = \vec{a}\Delta t$. After this, with no more acceleration, it continues upward at constant velocity and at some later time t is found at the point O''. The total time from starting the acceleration at O until getting to O'' took $t + \Delta t$ seconds. If one considers a sphere around O of radius $\gamma = c(t + \Delta t)$, then outside that sphere there cannot be any knowledge that the charge was moved, since that message travels with the speed of light c. So, the field in that region points radially outward from point O (**18:45**).



Once the charge is moving uniformly, the field is again radial. The corresponding field line (i.e. at the same angle as a line from O) can be drawn from O''. It points radially outward from O'' and extends out to a radius of ct from O', the point at which the motion became uniform. Since it is actually the same field line, the line inside the inner sphere must be connected to that outside the outer sphere by a line in the region in between (**21:00**). The portion of this line perpendicular to the radius has a length $\sim u_{\perp}t$, an approximate result since the charge was not traveling with speed u for all of the period t.

In this context $\perp$ means "perpendicular to r". If $u \ll c$ and $\Delta t \ll t$, then $r = ct \gg ut$. So, the distance moved along that leg of the triangle is very close to $u_{\perp}t$. Only the very small distance $O - O'$ is ignored. The radial thickness of the shell is $c\Delta t$ (**23:15**). An approximately linear segment joining the inner and outer parts of field line can be decomposed into parts parallel to the radius, $E_{\parallel}$, and perpendicular to it, $E_{\perp}$. This whole shell containing the kink travels outward at the speed c, and the field $E_{\perp}$ is very suggestive of the transverse field that we saw exists in an EM wave. If an observer was looking in toward where the charge accelerated, then this perpendicular E field would correspond to that of a traveling wave.

The task is now to calculate that perpendicular $\vec{E}$ in the shell (**25:40**). From the geometry, $\frac{E_{\perp}}{E_{\parallel}} = \frac{u_{\perp}t}{c\Delta t}$. Since $u = a\Delta t$ leads to $u_{\perp} = a_{\perp}\Delta t$, we find $\frac{E_{\perp}}{E_{\parallel}} = \frac{u_{\perp}t}{c\Delta t} = \frac{a_{\perp}(\Delta t)t}{c\Delta t} = \frac{a_{\perp}t}{c}$. We could get $E_{\perp}$ if we could figure out $E_{\parallel}$. Using $t = r/c$ gives $E_{\perp} = \frac{a_{\perp}r}{c^2}E_{\parallel}$ (**27:45**).



We can find $E_\parallel$ by considering Gauss' Law. One side of a Gaussian pillbox will be outside the region that knows about the change in the charge's motion, and the inside will have the $E_\parallel$ we want to figure out. There is no charge in the box, so the surface integral of $\vec{E} \cdot \hat{n}$ must be 0. The contribution from the sides is 0 since $E_\perp$ is the same on both sides, i.e. as much points "out" as "in", and $E_\parallel \perp \hat{n}$. Since the top and bottom areas are basically the same, $\vec{E}$ must be the same on both to give a surface integral of 0. The field outside is simply $\vec{E} = \dfrac{q}{4\pi\epsilon_0 r^2}$, and so, in the shell, $E_\parallel = \dfrac{q}{4\pi\epsilon_0 r^2}$. Substituting into $E_\perp = \dfrac{a_\perp r}{c^2} E_\parallel$, we have (**30:50**): $E_\perp = \dfrac{a_\perp r}{c^2}\dfrac{q}{4\pi\epsilon_0 r^2} = \dfrac{a_\perp}{c^2}\dfrac{q}{4\pi\epsilon_0 r}$. This is the classic derivation of the result already known in the late 19th century for the strength of the electric field perpendicular to the direction in which a disturbance is traveling. It is inversely proportional to r which is also a natural consequence of the conservation of energy. This is different from static electric fields, which fall off as $1/r^2$. Another difference from the static case is that we have to account for the time for the radiation to get from the point of emission to the point of observation. If the radiation is observed at time t, then in fact it was emitted at a time $t' = t - \frac{r}{c}$. Incorporating this time delay, the equation becomes $\vec{E}(\vec{r}, t) = -\dfrac{a_\perp(t')q}{c^2 4\pi\epsilon_0 r}$ (we will see the origin of the minus sign later). As before, we could calculate $\vec{B}(\vec{r}, t)$ and in turn $\vec{S}(\vec{r}, t) = \dfrac{\vec{E} \times \vec{B}}{\mu_0}$ (**33:50**). We must also consider that $a_\perp$ depends on the observation direction. The angle θ between the acceleration and observer is zero looking along the line of the acceleration, in which case there would be no perpendicular component. Looking in at 90°, $a_\perp$ is the same as a, the total acceleration. So the strength of the electric field is a strong function of θ.

From the original diagram, it can be seen that the kink is in the opposite direction to $\vec{a}$, which explains the minus sign noted above. The component $a_\perp$ is proportional to $\sin\theta$ so the Poynting vector is proportional to $\sin^2\theta$. The radiation has a strong peak of energy emission in the direction perpendicular to the acceleration and is zero along the acceleration. This wave is spherical, not a plane wave but far from the origin a plane wave solution is a reasonable approximation (**37:20**).

To summarize, for an acceleration $\vec{a}$ of a charge q, as observed at a point $\vec{r}$, the electric field $\vec{E}$ is in the plane of $\vec{r}$ and $\vec{a}$, $\vec{E} \perp \vec{r}$, $|\vec{E}| \propto |a_\perp|$, and $|\vec{E}| \propto 1/r$. If q is positive, then $\vec{E}$ is in the direction opposite to $\vec{a}_\perp$, and if q is negative, then $\vec{E}$ is in the direction of $\vec{a}_\perp$ (**39:15**).

The $1/r$ dependence is related to conservation of energy since the Poynting vector is proportional to E^2. The area of a spherical surface goes up as the r^2: if at the same time the Poynting vector goes down by $1/r^2$, the total energy flowing through any spherical surface will be the same.

The 80 MHz transmitter (wavelength 3.75 m), used last lecture to show the effect of polarization, is now used to see the effect of the angle θ. When the transmitting and receiving antennas are parallel the signal is largest, while when they are perpendicular (but, unlike before, still in the same plane) the light is not illuminated at all (**42:30**).

To calculate the total power over a sphere, the Poynting vector, with its $\sin^2\theta$ dependence, must be integrated over a sphere. The result is the power (energy per second in watts) over any sphere being $P = \dfrac{q^2a^2}{6\pi\epsilon_0 c^3}$ (**44:35**). This is known as the Larmor formula. It gives the power that must go into the electromagnetic fields when a charge is accelerated, in addition to the mechanical work that must also be done to accelerate the charge (due to its mass).

How do we in fact accelerate charges? The answer is a bit embarrassing. In almost all cases, we accelerate charges by exposing them to electromagnetic radiation. This seems like a "Catch-22" in that one usually uses EM radiation to do the acceleration to get EM radiation. Consider an electron exposed to plane EM radiation $E_0\cos\omega t$ producing a force of $F = qE_0\cos\omega t$ (**46:10**). We consider that the electron is bound, as in an atom, and in equilibrium, so that there is a restoring force. Ignoring damping, we can write that there is a resonant frequency $\omega_0^2 = k/m$, where k is an effective spring constant and m the mass of the electron. The equation of motion is $\ddot{x} + \omega_0^2 x = \dfrac{q}{m}E_0\cos\omega t$, which was seen before in the case of forced oscillations The solution is $x = A\cos\omega t$ with amplitude $A = \dfrac{(q/m)E_0}{\omega_0^2 - \omega^2}$. To calculate how much radiation is produced, we need to know the electron's acceleration, $\ddot{x}$ which is (**49:50**) $\ddot{x} = -\omega^2 x = \dfrac{-qE_0\omega^2}{m(\omega_0^2 - \omega^2)}\cos\omega t$.

The total power is proportional to a^2, and forgetting all the constants $P \propto \dfrac{\omega^4}{(\omega_0^2 - \omega^2)^2}$. If the motion is far from resonance such that $\omega << \omega_0$, this simplifies to the very famous result that $P \propto \omega^4$. This is known as *Rayleigh scattering*. It is called scattering, since the radiation coming out has the same frequency as the incoming radiation. There is no change in color but only in direction. For oxygen and nitrogen in the atmosphere, the resonance frequency is in the ultraviolet (UV), so the condition of being far from resonance holds. Blue light has a higher frequency than red light so is more efficiently scattered. Since blue light has a wavelength of about 4500 Å (450 nm) and red light about 6500 Å (650 nm), the ratio of their frequencies is about $\dfrac{\omega_{blue}}{\omega_{red}} \approx 1.5$. When raised to the fourth power to give the relative scattering efficiency we get $1.5^4 \approx 5$. Blue light is scattered about five times more than red light. For small particles in the atmosphere (a few tenths of a micron) this result holds, and for larger particles the effect is not very pronounced. By the time the particle size is 5 microns, all frequencies of light scatter about equally.

Another effect of Rayleigh scattering is that the light becomes linearly polarized, in fact 100% polarized for scattering at 90° (**54:00**). Consider unpolarized light coming directly out of the page. The incoming $\vec{E}$, and hence the acceleration of charges causing the scattering, is in the plane of the page but its direction changes rapidly. The outgoing $\vec{E}$ is in the plane of the acceleration and the line to the observer, with the latter also in the plane of the page for 90° scattering. Thus the $\vec{E}$ of the scattered light must also be in the plane of the page. Finally, $\vec{E} \perp \vec{r}$, so the outgoing $\vec{E}$ can only points in one direction, i.e. 100% polarized. (**56:55**).

This effect is demonstrated using cigarette smoke. In smoking a cigarette (not advised due to adverse health effects) combustion creates extremely fine dust particles, a few tenths of a micron in size and thus ideal for Rayleigh scattering. If such smoke is blown into a vertical white light beam, blue light will be preferentially scattered by these very small particles. White light contains all colors, but since the blue is better scattered, the smoke will appear blue from the side. In the demonstration, the angle of scattering out into the audience is close to 90°. Thus the scattered light is expected to be linearly polarized in the horizontal plane. To get a lot of smoke, multiple cigarettes are used! The light does appear appears bluish and is also shown to be polarized (**1:00:30**).

If the smoke is held in the lungs, the water vapor naturally present there will precipitate onto the smoke particles, making instead small water drops which are too large to do Rayleigh scattering. We expect the scattering of the light to instead be white. This is demonstrated and an "instant replay" (**1:02:35**) most clearly contrasts the cases of smoke and the larger condensed droplets.

The sky is blue due to light scattering from very fine dust and even density fluctuations between molecules. In clouds, however, there are water droplets which are fairly large and scatter all colors of light without much preference, thus appearing white. If you are standing on Earth with the Sun quite high in the sky at midday, then in a direction away from the Sun, the sky is blue. If one looks 90° from the Sun, the sky will be 100% linearly polarized.

The sky scatters about 1% of the sunlight, more when the Sun is lower in the sky due to the longer path in the atmosphere. Therefore, a setting Sun (or Moon or planet or stars) appears red (**1:05:20**). Anything lit up by the Sun's setting or rising beam, for example a cloud, is also red. The more fine dust is in the atmosphere, the more vividly colored are sunsets (for example after injection of volcanic dust or pollution). Example illustrations are also shown for stars in the Pleiades cluster, an astronaut walking on the Moon, and Aerogel. The latter is a material which has only four times the density of air and in fact is 99.8% porous, with very small silica particles, of 1-2 nm size (much smaller than the wavelength of light). Light passing through aerogel appears bluish while its shadow, contained the remaining light that was not scattered, is reddish.

In a total lunar eclipse, when the Moon is completely in the shadow of the Earth, it appears red in color (**1:10:00**). When the Moon covers the Sun as seen from Earth in a total solar eclipse it just covers the Sun. Since the Earth is four times larger in diameter than the Moon, when the Earth covers the Sun as seen from the Moon (causing a total lunar eclipse as seen from Earth), it is totally covered, but the scattered sunlight from the atmosphere is seen around the edge. This sunlight hitting the Moon's surface has traversed a large length of atmosphere, so it is red in color. From the Moon one would see the Earth, four times larger than the Moon would be in our sky, and very black with the Sun behind it, but surrounded by a beautiful red ring of atmosphere.

Finally, "the mother of all demonstrations" is done (**1:12:00**). It will simulate a blue sky, a red sunset, and the linear polarization at 90° from the Sun. Sulfuric acid is added to a container with sodium thiosulfate dissolved in water, causing a chemical reaction which will precipitate small particles of sulfur. Initially, these are very small, less than a micron in diameter. The white light from a projector will be scattered from the clear-sided container out into the audience's direction. Initially, very little goes in that direction since the sodium thiosulfate solution is very clear. When the reaction occurs, and small sulfur particles are in the water, Rayleigh scattering will cause blue light to go out toward the audience. Those for whom the scattering angle is 90° will see 100% polarized light, in the vertical direction. Those in other directions can expect to see partially polarized light. As time goes on, there will be more and more sulfur, and more scattering, so the transmitted light will become redder. After the sulfuric acid is added, some blue scattered light is seen and a polarizer rotated in front of the container shows a high degree of polarization (**1:15:30**). The "Sun" of the transmitted beam appears to steadily become more reddish. Over time, the scattered beam gets brighter, the "Sun" gets redder and clouds appear (likely due to irregularities in the chemical reaction). "Sunset" is simulated by moving the projected beam downward to simulate a horizon, to the laughter of the audience.

8.03SC Physics III: Vibrations and Waves
Fall 2012

These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.