PROFESSOR: Welcome back. And today we will continue doing some more problems to do with standing waves. It's a very important topic, so I thought it worthwhile considering some different kinds of problems.

So the first problem I want to discuss with you is the following. Suppose you have two equal-length strings, each of length L, but they are very different. One is very massive. The other one is very light. And we tie the massive one at one end. It's connected to the other one in the middle. And the other one is tied here.

The whole system is taut. And the question is, what are two lowest normal modes of vibrations? Calculate the frequencies of these and sketch what they look like.

And I like this problem because it's counter-intuitive. You could take a few seconds for yourself and think whether you could predict by guessing. And I'm almost certain your guess will be wrong. Certainly, the first time I tried to figure this out, my intuition gave me the wrong answer.

OK. Now, again, it's idealized. We'll assume that these strings are ideal, lossless strings. All the displacements are very small. So in the derivations of the equations of motion, et cetera, the usual sine theta equals theta, there is a constant tension in this string T. Because they are connected, of course, like this, the tension everywhere must be the same.

And all we know is that the density per unit length of this string, which I call mu sub H, this is the heavy string, is much bigger than the density, the mass per unit length, of the light one. Now, the fact that they don't give me the values of these, just say this, basically is telling me they want almost a qualitative answer. I cannot calculate it exactly if I don't know these two numbers. All right?

Now, the other thing we are told is, of course, a system like that can oscillate in three dimensions. It could oscillate up and down, out of the board, in, et cetera. They're telling us limit ourselves to oscillations of the string in the plane of the board. OK? So it's a 2D problem, not a three-dimensional problem.

OK. So how do we do this? We immediately recognize that when you have a continuous line of harmonic oscillators in the continuum limit, and that's what the string is, the equation of motion of the string is this. This is the standard wave equation. It's the same equation for the heavy string and the light string.

And the only difference is the phase velocity v, which depends on the tension and the mass density of the string. So the heavy one we will call v sub H, and it's the square root of T over mu H. And the light one I call v sub L, square root of T over mu L.

OK. Since we know that mu H is much bigger than mu L, we immediately know that vH is much less than vL. That's the information we have about those strings. OK.

What else do we know about this system? I mean, we are now in the usual process of translating a physical situation into the mathematical description of the system. OK? In this particular idealized case, this is almost a mathematical description up there, but still. So this is the equation of motion.

And the other thing we know about the system are the boundary conditions. We know that at both ends, at x equals 0 and at x equals 2L, the string is tied down. So that for all time, therefore, the displacement will be 0, y equals 0.

We know that at the boundary, the strings are connected. So at all times, y of H must be equal to yL. Otherwise, it would be broken. So they're connected. The displacement of the string in the middle, both the heavy one and the light one, is the same, and this is true at all times.

And the other thing is that the slope of the string on both sides must be the same. How do I know that? Well, imagine the junction to be some object. Then that junction has 0 mass. If it has 0 mass, you cannot have a net vertical force on that. You cannot have a net force on it, period. Or because of Newton's laws, force on the 0 mass object will give it an infinite acceleration. It will just disappear there.

So at the junction, there can be no net force. The tension on the string on both sides is the same. And therefore, the slope of the strings at all times, on the left it must equal the right.

So this is the complete mathematical description. And we are asked to, with this information, figure out, what are the two lowest mode frequencies and what are their shapes? OK? So that's what now I will do.

OK. Now, we know that what we're interested is in the normal mode. If you have a normal system oscillating in the normal mode, you know that the all parts oscillate with the same frequency and same phase. That's the meaning of being in the normal mode. OK?

So we know that when this system is oscillating in the normal mode, both sides of the string will be oscillating with the same omega, same frequency and phase. All right? And also, on both sides we have a string, a row, a continuous row of oscillators, and we know what the normal mode oscillations of such a system is. We've seen it over and over again. It is sinusoidal in fall.

So the general expression of the displacement of the string, whether it's on the left side or on the right side, will be the displacement of it from equilibrium is a sinusoidal function in position multiplied by a sinusoidal function in time. This function will be different for the left and the right, the spatial distribution. This part will be the same.

Why would this be different for the two sides? Well, this equation satisfies the wave equation. It is the solution. I am looking for that wave equation. If this is to satisfy the wave equation, you'll find that omega/k, which you can write like lambda times f where lambda is the wavelength and f is the frequency-- this is the angular frequency on the wave number-- has to be equal to v. Otherwise, this would not satisfy the wave equation. OK?

And v is different for the two strings. All right? We showed that. We discussed it a second ago. v, the phase velocity for the heavy string, is much smaller than for the light one. And therefore, if omega is the same, the k would be different in the two cases. OK. Fine.

Now, using this, we know that the wavelength is given by v/f. Just I've copied it from there. And since the frequency on both sides is the same but v is different, I can immediately conclude that the wavelength of the part of the string which is massive, the heavy one, over the wavelength of the string on the light side is simply equal to vH over vL.

But we've been told that the density of the heavy one is much greater than the light one. Therefore, the velocity, this one, is much smaller than this one. And therefore, we immediately can conclude that the wavelength in the normal mode oscillation of the heavy one is much shorter than of the light one. All right.

So now if we were given more information exactly what these quantities are, et cetera, we could write the equations of the string on both sides and try to solve it. And in the next problem, I'll try to do an example which is exact. Here I'm treating the problem of it more qualitatively. And let's see what we can figure out just from this fact that the wavelength of the massive one is much smaller than of the light one.

We are now looking for a solution to this problem where everything is moving with the same phase and frequency. All right? And we are looking for the lowest normal mode, in other words, when everything is moving as slowly as possible subject to the boundary conditions.

Now, moving slowly means big wavelengths. So what we have to try to figure out-- here is a sketch of this string. This is the heavy one. This is the light one. And what we want, we want to find, on each side, this string is just like any old string.

In the normal mode-- in a single mode, will have a sinusoidal function, all right, of a single wavelength on both sides. So this must be a part of a sine curve or a cosine curve. So must this. And we want to make it as long wavelength as possible, and we want to satisfy all the boundary conditions.

So here it's located. Here this one is located. OK? The one on the right, you want to make as long a wavelength as possible, and so this is almost a straight line. This is part of a sine curve, but it's almost a straight line. This one is part of a sine curve. And at the boundary, we must have the two touch, so the string is not broken, and the slope to be the same.

So the answer is obvious. The solution will be something like a sine that just turns around near the top here. And when the curvature of this sine curve is such that it points towards this point, then we have found the lowest frequency solution. All right? This is approximate. This will have very slight curvature but approximate.

And so what we see, when the wavelength on the left-hand side is basically 4 times this, which is 4L. So this will be the lowest normal mode that we figured out without solving any equations, from just our knowledge of the boundary conditions and what normal mode solutions are on the string like this. All right?

Now, I can now, knowing the wavelength, I can calculate the frequency this corresponds to because we know that-- we've got it here-- omega is simply k times v. All right? k is 2 pi over the wavelength. So we have omega 1. This is the lowest normal mode. Angular frequency, 2 pi over lambda 1 times vH.

We know what lambda 1 is. We just argued ourselves to show it's approximately 4L. It's actually a little less than 4L, but approximately 4L. And so this is the lowest normal mode frequency.

All right. Now, what's the next normal? They ask us for the two lowest normal modes. We repeat the argument. We are now trying to figure out-- each of these strings now we would like to oscillate with slightly higher frequency. This, again, will be a sine curve on both sides.

And again, we've got to find a situation where the curvature-- the sine has shorter wavelength-- but the minimum shorter that I can make it so that I satisfy the boundary conditions as before. The strings must not break, and the slope must be the same.

Well, if you play around, you'll see that the next lowest normal mode will look like this. The string goes like this. All right? And here, the massive one is just curving. And when its curvature is such that it points towards this origin, where this is almost straight, that will be the sketch of the normal mode. So that's what it will look like. That will be the next lowest normal mode.

Now, what is the wavelength? You can calculate that from this length. It's just a little less than 4/3 L. OK? And again, like before, since we know the vH and the wavelength, we can calculate the frequency. So these are the two normal mode frequencies.

And why did I say it's counter-intuitive? For you, maybe it is intuitive. But I would have thought that if you connect a heavy string to a light one, it's the light one that would be wobbling up and down and not the heavy one. And in the reality, it's the opposite that happens.

It's the little guy that seems to be deciding what the big guy is doing. And this is the one which is-- also this one-- all the time is nearly straight. The role of this one is simply to keep this end of the heavy string tied down, and it determines the boundary condition here. OK? But it's an interesting case to think about.

OK. So that's the first problem I wanted to do. Now let's go to another one. In some ways, it's similar to the one we've done. But just in order to show you how general these discussions are of these problems, I'm taking a situation where the displacement from equilibrium is not transverse to the location of the oscillator. I'm going to consider an example of longitudinal waves.

Let me discuss the problem here. This is the problem we're going to consider. We're going to consider a solid rod attached to a wall. So this is just a round rod. It is connected to another rod. Let's say take some case, maybe a tungsten, a lead alloy, a rod of tungsten lead, which has a certain Young's modulus y.

And Young's modulus, let me remind you, it is the extension per unit length if you apply to it a force, certain force per unit area. So it's extension per unit length divided by force per unit length. It tells you how, under the system, what strain you get into the system.

This is analogous to like, in a spring, we talk of the constant k for the spring. Tells you how much the spring [INAUDIBLE] when you apply a force. So the y will tell you how much this extends under a force per unit area.

And I'm going to assume this rod has a density rho, mass density rho. It's connected to another, but this time much lighter rod, a rod which has three lengths, 3L length, same area. They are welded together here, connected together, glued together. OK?

I've chosen two materials. And if you look in the books, you can find different materials having same y. This is a carbon fiber composite. This is a tungsten-lead alloy. And they both have the same Young's modulus. But this is much, much lighter. The density of this is 1/9 of the density of this material.

And I chose these-- I wanted to come up with numbers where one can solve the problem completely. In the previous problem, I want to do this qualitatively so I didn't have to pick up numbers which gave you a final mathematical description which has an analytic solution.

OK. So we have these two rods connected like this. This one is attached to a wall. It cannot move. And this is free. I purposely wanted a problem where there are new things coming in so that you can see how one tackles it. OK.

As usual, there's some assumption. We're going to assume that this is a lossless system, while the rods are oscillating that they're not losing energy, that the oscillations are small so that we can do all the usual approximations.

We'll make the assumption that this radius is small compared to this length so that we don't have to worry about transverse oscillations of the system. We're only talking about longitudinal oscillations. So we're talking about small oscillations in the longitudinal direction.

And we are told that for such a system-- and you could derive it for yourselves-- that the phase velocity is the square root of Young's modulus divided by the density. So it's different for the two sides. OK?

And the question is, can we calculate for this-- knowing all these quantities, these are now all given quantities-- can we calculate what is the angular frequency omega 1 of the first normal mode as this oscillates? The problem, in some ways, is similar to the previous one.

But as I say, normally we more often do things like strings oscillating because it's easier to plot on the board the displacement of the string, which is in the y direction, and the position in the x direction.

Here we have the displacement from equilibrium at any point in the same direction, in the same as the position, and that makes it harder to plot. That's why one normally doesn't take this example. We tend to take examples which are easier to illustrate on the board.

OK. So let's now, as usual-- this is the physical situation-- what does it correspond to as a description in terms of mathematics? Well, let's define, at any point x and time t, the displacement of the material of the rod from the position of equilibrium by the Greek letter psi. So it's psi at x of t

Now, this system is, again, an ideal system of a continuous distribution of harmonic oscillators. Each piece of mass oscillates backwards and forwards longitudinally. That's the oscillator. It's continuous, that is, an oscillator at every x. OK? And each oscillator is coupled to its neighbors. So this is mathematically exactly the same situation as a string, where the oscillators are there oscillating up and down instead of longitudinally.

So the equation of motion of this system will be, as always, the wave equation. This is the longitudinal displacement, how it's oscillating. v is the phase velocity. And we know that the phase velocity is the square root of Young's modulus over density. Since we know the ratio of the densities, and the y is the same for both materials, you know that the ratio of the phase velocity in the two materials, v2 over v1, will be 3, the square root of 9. OK?

And as always, we know that omega/k is the phase velocity. Otherwise, it wouldn't satisfy this equation. So we know that the k2/k1-- and I'm reminding you, k is 2 pi over the wavelength, right? So k2/k1 is 1/3. OK. So this is the equations of motion and the constants of it.

What are the boundary conditions? Well, they're slightly different to the one we've had of strings attached at both ends. We know that, in the middle, the two rods are attached to each other. So the displacement from equilibrium, whether you're on the left rod or on the right rod, will be the same. So the one on the left, which I call psi 1, at position L [INAUDIBLE] t, will be equal to psi 2 at L of t. So that's one boundary condition.

Another boundary condition is the one analogous to when you have two strings joined, that the slope of the strings must be the same on both sides because the junction has no mass. Similarly here, the junction between the two rods has no mass, so there cannot be a net force on that junction.

And since the y, the Young's modulus, is the same, the tension is the same inside, then the slope of d psi 1 dx at that junction must equal to d psi dx at the ther.

As I repeat, this is analogous to a junction between two strings when you're talking about transverse oscillations of strings. The slope at the junction must be the same on both sides, unless there is a massive object at that junction. And we don't have such a thing. If there was a heavy bead or something at that junction, this would no longer be true.

The other thing is, what are the boundary conditions of the ends? On the left end, it's easier. We said the left end is attached to the wall, therefore it cannot move.

OK? Therefore, the displacement from equilibrium of this left end is equal to 0 at all times.

The right end is different. It's free, and so it's certainly not 0. What do we know about that? Well, at the end, the rod ends, and there is no mass or anything there. So the end of the rod is not pushing anything. There's effective, it's pushing a 0 mass at the end of it.

Therefore, there cannot be a net force on that. And so the rod cannot be compressed at the end or elongated because then it would be exerting like a spring attached to nothing with exerted force on it, which is not possible.

And so the rate of change of psi 1 with x of the displacement from equilibrium from x at the-- sorry, I'm here, I'm here. The slope of psi 2 with respect to x at the end, which is-- I am sorry. This will be 4L. This is my mistake. This is 4L here. OK. At the end of the rod, it has to be 0.

OK. So this is now the mathematical description. And all we have to do is solve the equation of motion subject to these boundary conditions and find out what is the lowest frequency that satisfies the boundary conditions and the wave equation. OK?

And again, I have written out the solution for you. And let me go a little faster now. We now have experience. We know that the wave equation has standing wave solutions, which are some amplitude times a sinusoidal function in x and a sinusoidal function of time.

And now to save time, I am going to inspect immediately the boundary conditions, and using, in particular, these boundary conditions, the ones at x equals 0 and 4L for all time. And I'm going to figure out sinusoidal functions of x and t, which satisfy these boundary conditions.

Now, we know that we are talking about standing waves. So for both rods, the angular frequency, the [INAUDIBLE] oscillations in time, will be the same. So this part will be the same. This part must satisfy the boundary condition at the extreme left and right, which forces me to write here a sine k and here cosine k2 of what happens at the boundaries. OK?

A good exercise for you is slowly think through the boundary conditions and the general solutions, the normal mode solutions, of the wave equation. And you'll come to the conclusion that this describes the standing waves in the two rods.

OK. We know further that omega is equal to k1 v1 in the left rod, and equal to k2 v2 in the right rod. This has to be the same in both, as I said, because we are in the normal mode. We're in a normal mode, where the whole system is oscillating with the same frequency and phase.

OK. Now, we will use the other bit of information, the boundary conditions at the junction, which I've written here. And from that, try to find a solution that satisfies these and see what is the angular frequency of oscillation for that.

So we can now write at x equals L-- these are the two equations. We'll take the displacement of the rod on the left and make it equal to the right and the slope on the left equal to the right. On the left, this is the equation. On the right is that.

At that boundary to make the two equal, I get that A1 sine k1 L at x equals L is equal to A2 cosine times this, all right, at x. When we're talking about x2 equals L, L minus 4L is minus 3, et cetera. OK? The cosine terms cancel because the time part cancels because they're the same on both sides.

So this is the equation we have to satisfy so that the string is not broken. And I'm just writing it like this because we know that-- we did that earlier-- that k2/k1 is 1/3. And the cosine of a plus and the minus angle is the same, so I've just replaced that with plus k1 L.

This is now looking at the slopes of the displacement psi at that boundary. So I differentiate these with respect to x, all right, this one and that one, and equate them. And I get this equation, all right? And so in order to satisfy the boundary conditions and the wave equation, I end up with these two equations.

I've chosen things such that these can be solved analytically. It's not hard. You can do it for yourself. You have two equations, two unknowns. The unknown is the ratio of A2 to A1. That's one unknown. And the other unknown is k1.

You can solve these equations trigonometrically only because I've chosen numbers of rho, et cetera, which make the answer come out. And you end up that you get k1 from it. Knowing k1, you can calculate omega. And this is the answer you will get. I won't waste your time here. You can do this algebra for yourselves.

So here I've got the final answer because I had all the input. If I had tried to do the same with the string one, if I chose some random ratio of the density of the string on both sides, everything else would have been similar. But here I would have ended up with a transcendental equation, in general, which I cannot solve analytically.

And so if you take any random case for yourself and try to solve it, you would find that, at this point, you would be stuck. You would be stuck with a transcendental equation, which, of course, you could solve numerically over the computer, et cetera. OK. So that's the end.

And now I'll come to the third, yet another example of standing waves, this time in three dimensions. So now we come to the third standing wave problem. And just for variety, I decided to take one in three dimensions. So the problem is the following.

Suppose you have a room, which is 2 meters in size by 3 meters by 4. What is the lowest standing wave frequency that you can get in such a room? OK? So it's basically a problem to do with pressure waves, because that's what sound is, standing pressure waves in three dimensions.

We are told that, in this particular room, what the density of the air is. We're also told what the bulk modulus, in other words, how compressed the air can get and what's the resultant pressure-- what kind of compression gives rise to air. And we are asked to calculate lowest normal mode frequency of this system.

OK. So how do we do this? Not surprisingly, you'll remember already when we were talking about one-dimensional system, way the standing waves of strings or on a rod, et cetera, it's quite complex. If you have three dimensions, the situation is getting more and more complicated.

In this course, we don't actually derive the three-dimensional wave equation. But by analogy with the one-dimensional one and two-dimensional one, we sort of indicate what form you would expect this to be. And it actually is the case.

The three-dimensional wave equation for a scalar quantity like pressure that has no direction, all right, is written here. It's the d2p dt squared, like for the one dimension, is equal to v squared times the gradient squared of the pressure, which, if I write it in Cartesian coordinates, it's written here.

Clearly, if I take a three-dimensional system but reduced two dimensions with some small distances so that, in essence, it's a one-dimensional problem, this gives you the wave equation you would expect. If I remove two of these terms, what I have is the wave equation in one dimension.

OK. In this equation, p is the excess pressure of the air above its equilibrium Imagine the room. Everything is at equilibrium. The air is not moving. The pressure is not changing, et cetera. There will be some pressure.

What we are interested in is what happens if we displace that air from equilibrium, change its pressure, and let go, what will happen? And so it's that excess pressure we are interested. That's the p.

And we are interested to see if there is a way to do that such that the air in the room everywhere, the pressure, oscillates with the same frequency and phase. If it does, then in that room the air will be in its normal mode, oscillating in its normal mode.

One can show when one derives this-- and I did not derive it, of course-- is that this v is the phase velocity of propagation of pressure waves in air, the sound velocity, which is the square root of the bulk modulus divided by the density. OK?

OK. So the translation of this problem is this is the equation of motion of the system. And what are the boundary conditions? The boundary conditions are that at the edges of the room, at the walls, the pressure does not change perpendicular to the wall.

So for example, the two walls, at the end in the x direction, at the end there, at one side of the wall, the rate of change of pressure with position and at the other, it will not change. Top and bottom, perpendicular, it will not change this way. It will not change.

Why is this the boundary condition? The reason for that is that if you imagine at the wall, the molecules cannot move, right, because of the wall. So therefore, you cannot there, near the end, increase the number of molecules.

So perpendicular to it, that the number will be independent of the position, which is dp dx, is equal to 0. OK? And this is analogous to the string situation, which is attached to a massless ring where the slope is constant at the boundary. OK.

So with these boundary conditions, what are the solutions of this equation? OK? To find the most general solution of this equation, it's obviously very, very complicated. There are infinite possibilities. But from my experience of what happens in one dimension, we can do a pretty good, intelligent guess. And what I've written here, we can guess that the solution will look like that.

Why is it like this? Well, first of all, we want a solution of this equation, which is a normal mode, meaning everything oscillating with the same frequency and phase. And so independent of where you are, the time dependence must be the same for everything or else it would not be a normal mode. So that's why that. There is some amplitude, so that's p maximum.

You would expect symmetry between the three directions. There's nothing special between them. And you want some function here which satisfies the boundary conditions. And what you will find is that this, a cosine kx, the slope of this with respect to x gives you a sine. And at x equals 0, you will get the boundary condition satisfied. And we'll see in a second what constraint the boundary puts on this at the other end.

For this to be a solution of our wave equation, we must satisfy this and that. And so in some ways, you could say, look, I've guessed it. But then I'll use the uniqueness there, say, OK, I've guessed it or I knew it. I guessed this was the solution.

I will now check, does it satisfy everything I know, the wave equation and the boundary conditions? And the answer is yes, it does. And then I use the uniqueness there that says therefore that is the only solution that satisfies everything I know about the situation.

OK. Having done that, to get this I used the boundary condition as x equals 0. But I want this slope also to be 0 at the edges of the room. And that gives me a constraint on what each one of these cases can be. And you find that kx has to be some constant pi over Lx, where Lx is the length of the room in the x direction, so that the slope of this at the edge of the room is 0. Similarly for ky and kz.

And so you find that these L's, m's, and n's, have to be 0, 1, 2, or 3. They cannot all be 0. If they are all 0, then nothing goes on. This means that k is 0, which makes omega 0. It's a trivial, uninteresting solution to our wave equation. So at least one of these has to be non-zero.

So which one will give us the lowest frequency? All right? And so it will be the one where the k, all right, is as small as possible, which means that the wavelength is as long as possible. OK?

So if we make the L's and m, which correspond to the room, the 2 meter times 3 meters, those to be 0, the only direction we accept is in the direction of z, which is the longest dimension of the room, then that will give us the longest wavelength and therefore lowest frequency. OK?

So k, the largest value of the wavelength, will be when kz is pi over 4m, in other words, the wavelength in that direction being 8 meters. And since we know that v is the square root of the bulk modulus divided by the density, I could take the numbers that are given, I get that the phase velocity, or the velocity of sound in this room, is 342 meters.

The frequency, of course, is equal to the phase velocity over lambda. Or the way I normally remember, it's frequency times lambda is the phase velocity. All right? And I found the longest wavelength is 8 meters. So from that, I get that the lowest normal mode will oscillate at almost 43 hertz.

I'll just sketch for you here what we've essentially-- we've got this room, and we are looking for solutions which have the longest wavelength. See, in principle, one could have a wave like this in this direction. This, the slope, has to be perpendicular at the boundaries. You could have, in this direction, for example, a wave like that. And in this direction, you could have [INAUDIBLE] or a higher moment.

So it's hard for me to draw it. But you can imagine this pressure changing in all directions. But it has to be represented, as I say, by a sinusoidal function whose slope at the boundaries is always perpendicular like this.

And the wave number is, of course, the sum of the wave number of this squared plus this squared plus this squared, taken the square root over. The lowest frequency will be when, in this direction and this direction, it's straight.

There's no change of pressure, no change in this direction. And the only direction is in this direction, and this is the 4-meter direction. And that corresponds to a wavelength of 8 meter, which I calculated there.

And so this will behave in the same way as a one-dimensional system. Nothing changing in two dimensions. And only in one dimension it's changing. That will give you the lowest normal mode.

And the other normal modes, you can calculate the frequency as you change the little l, m, and n, play around. Always remember, at least one of them has to be non-zero. And you can calculate all the frequencies like this.

OK. That's enough for today on standing waves. Thank you.