PROFESSOR: Welcome back. Today, we will consider problems where the solutions are standing waves. So the first problem I'm going to do is shown over here. What we have is a string attached at both ends. The length of the string is L. And it is distorted in this strange shape. The dimensions are given everywhere. It's symmetric about the middle, OK?

Somehow miraculously, it's held fixed. You could imagine maybe sort of nails nailed here and here and fingers holding there. And then miraculously, we let go, all right? And the question is after we let go, what is the wavelength of the lowest mode that is not excited, the amplitude of the lowest mode which is excited? And finally, the question is, what is the shape of this string at this strange time?

Now, then what do they tell us? That this is an ideal lossless string, which has a mass per unit length of mu. It has a uniform tension T. For all the equations to work and the derivation of the wave equation, you have to assume that the angle is sine angle, et cetera, is equal like this, which is-- you look at this and say, holy smoke. How can this be the case?

Well, of course, this is an idealized situation. In reality, this would be very, very small compared to that, all right? And one would have to have some curvature at these points. The differences will be, the real situation from this idealised one, will only differ in some very high frequencies. So the first approximation, one can almost imagine that it's possible to set this up.

And this assumption, as I told you, initially is the system, at t equals 0, is stationary. OK. The above, as I was just saying, is really highly idealized. It is almost a mathematical representation already of a real situation. So you could imagine this almost being a mathematical description of this situation.

And what makes it physics rather than the mathematics is that string obeys the wave equation. It obeys this wave equation. Once I've told you that that picture on that diagram satisfies the wave equation where V is a constant given by this quantity, we've now converted this problem completely to mathematics.

One thing to notice, this is the phase velocity of propagation of a progressive wave on a string. So here, that time, they ask us in part C-- is the shape of the string at a time L over V from time equals 0. So we're now solving this problem, mathematics problem.

We know that if you have a coupled oscillator, there are normal mode solutions, solutions where every oscillator is oscillating with the same frequency and phase. For a system, a continuous system like this is, an infinite number of identical oscillators in a straight line for a string like that, the most general solution will also be-- one will be able to represent it as a superposition of normal modes.

Of course, if you look at such a system and you distort this from equilibrium, you can set up an infinite different varieties of solutions. But what I'm saying is that any one of those solutions however complicated, one could always reduce it to a superposition of normal modes. In particular, if you study the solutions of a system like this, one dimensional system like that, you will find that the normal modes are, in fact, standing waves which are sinusoidal.

So I've written for you here a completely general solution to a string. And you can write it, as I said, as a superposition of an infinite number of normal modes, pretty complicated. It tells you at every position on the string and at every time what is the displacement. You can calculate the transverse velocity by differentiating, et cetera. It just tells you everything about the oscillations of a string.

This is not any old string. Because we are told some things about it. We are told it's attached at both ends so that string has no displacement in the Y direction at x equals 0 and at x equals L. For all times t this equation will satisfy that string--

doesn't matter even what distortion it is-- provided we put some constraints on these various constants such that this is true.

Well, you can almost do it in your head. If you look at this, all these-- if for all times, at x equals 0, y is 0, this term, B of N, has to be 0. So this, you could forget about. And you're down to this. Furthermore, you know it has to be 0 at x equals L. That puts a constraint on k of n.

So your solution then is of this form. This is any arbitrary constant. For the wave equation, you can show for a string that these two are related through the phase velocity. And so once you've established this, then we've got that constant. So you're still left with quite a number of constants and an infinite series where the An can be anything. This phase can be anything.

It's always useful to check yourself. There are some things you can check. So for example, let's consider the n equals 1 harmonic, that normal mode, the first normal mode. If you look at this, it's clear that the wavelength of that harmonic is 2L. If you look at this, the period of that harmonic is 2L over v.

Period is 1 over frequency, or frequency is 1 over period. So the frequency is v/2L.

We know that for harmonic waves or standing waves that lambda times F is the phase velocity. Let's check it here, multiply lambda by F, and you get that this is satisfied. So at least this is a quick check that we haven't made some stupid mistake.

All right, so this satisfies any string of length L which is tied at both ends. But our string is not any string. We've said that initially, at t equals 0 for all positions, the string is stationary. So our description of it must take into account the equation which describes this shape of the string and it must have this boundary condition, OK? So we take this equation-- so far, we've reduced it to that-- differentiate it with respect to time, get this. And that's an easy-- we just have to differentiate-- the cosine gives you minus sine.

OK, so you end up with this. And this, we know is 0. It's 0 at-- I'm sorry, this is a t equals 0. I'm sorry. We know that at t equals 0 at every position x-- we said that. We're holding this string with nothing moving. At t equals 0, it's stationary. We then let go. So this is 0.

And if you look at this, this will be 0 at all values of x at t equals 0, only if all values of phi n is 0. So notice, gradually, by making use of all the information we have about the situation-- what string it is, where it's attached, how long it is, what is its motion at t equals 0-- we're gradually getting rid of the constants or determining them.

Now that phi n is 0, I go back to this and rewrite the equation. And now, we've ended up that the general equation which describes a string which has length L tied at both ends, stationary at t equals 0 is this. An is still to be determined. We have not made use, so far, of the information what is the shape of the string at t equals 0. That's what we'll do next.

But before I do this, I just want to make a comment. In many books, problems, et cetera, you'll find people starting with this equation. And you'll be wondering, how on Earth-- is this the most general equation for string problems? No. The answer why they started with this is, often without telling you, doing all this analysis in their head. This is only true for a situation where this string is tied at both ends and initially stationary. That forces you to that.

And An now is now not determined. And in order to find out the value of An-- in other words, the amplitude of the different normal modes-- we have to make use of the shape of the string at t equals 0. And that's what we'll do now.

All right, so now, I'm going to make use of the last bit of information I have. And since that defines the physical situation completely, there'd better be enough to find all the constants. There should be nothing left. They should be then completely determined.

So first of all, I said we didn't make use of the shape at t equals 0. At t equals 0, cosine of 0 is 1. So what we know is that the string shape is given by, from here, by this equation. So this tells us that-- this is a description of the shape of the string in terms of an infinite series of sinusoidal functions.

OK, we notice this tells us the shape of the string at every X position at t equals 0. So I look at that, the original shape. And I can describe it mathematically. y of x at time equals 0 is 0 at all points from one end of the string to where the distortion starts.

And it starts at 3/8 L if you look at the diagram. Then from position x of 3/8 L to x is 5/8 L, the string is straight again, and its displaced by a distance H. And afterwards, past the 5/8 L, it's back to 0.

For a second, let's go back and just look at that. So we're talking about this shape, right? It's 0, 0 in Y, and then jumps to H between 3/8 L and 5/8 L. This shape, when you let go, that will determine the amplitude of the different normal modes that are important in the oscillation of that system.

OK, so we must determine these infinite number of constants. At first sight, it looks like an impossible task if it wasn't for a brilliant mathematician in the 18th century, Fourier, who made a relatively simple but incredibly important observation. When applied to this problem, it is the following. He realized that one can do the following trick.

If you take this function, sine n pi over L x to 1 which is describing our shape over there, if you multiply it-- and by the way, n-- I'm just reminding you-- goes from 1 to infinity. It's 1, 2, 3, 4, 5, 6, et cetera. If you take this function and you multiply it by a similar function, but with different value of m, then this integral from one end of the string to the other, is equal to 0 if this, the m you've chosen here, is different than n.

Try it. I don't want-- a good exercise for yourself. Take this function. And do the integration, and you'll find it comes out to 0. It's not hard to do. So that is 0 if n is not equal to m.

On the other hand, if m equals n-- so this is the integral of sine squared n pi over L-- it comes out to be L/2. You can do it for yourself. This trick allows us to solve for all values of An.

Now, let me tell you, I did not do this in-- the Fourier trick-- in general. If you go into books, you'll find the formulation of this trick known as Fourier's theorem for any periodic function. The advantage of learning it, how to do it for any function, you don't have to do this integral every time. Because suppose our function turned out there to be a cosine, we would have to then figure out, what do I multiply it by to make this come out so simply?

As I say, the advantage is then you can use formulate in books, et cetera. There's nothing wrong with that. It is good. And the way one changes a physical situation into one where you can apply Fourier's theorem, which only applies to periodic functions, is to take your original shape, and out of it, create a periodic function. Once you've done that, you can use Fourier's theorem.

I've, in this particular case because I find it easier to explain it, simply essentially derived the Fourier theorem for this particular problem. For this particular problem, the function by which you multiply this is sine m pi.

OK, once you've realized that this is true, I can go back and use it to determine every An. I go to this function. I multiply both sides in turn by sine m pi over L for every n where I want to find the value of An, for each one. So I have to do an infinite number of integrals in principle. But often you don't need to know every value of n.

You may want to know each one of An is the amplitude of the harmonic. You may be interested in what is the amplitude of the first harmonic solution or the second or third. However many you want, so many integrals you have to do. No magic. You're basically solving infinite number of equations.

And if I apply this to our problem, we find that An is equal to 2/L. It's from here. The integral of 0 of 2L of y sine at dx. Right? All I'm taking, I'm multiplying both sides of this equation by sine of m pi L over x and integrating them both from 0 to L. On one side, you end up with An and on the other, 2/L times this integral. Then once you've done this integral, you know the answer.

In our case, y of x of 0 is particularly easy. It's 0 in this range, 0 in this range, and a constant for the rest. Well, over the part of the range where it's 0, if I multiply something by 0, it's 0. And so I only have to do this integral over the range where y of x for 0 time is H. So I have to do this integral. And now I have reduced the situation that every n that we are interested in, we can calculate by doing this integral. So in principle, we've solved it for all values of n.

So finally, I can answer the question. So the first question was what lowest value of n for which An is 0.

Well, let's start. Take number 1, 2, 3. And for n equals 1, this integral is not equal to 0, because what you have to do is you basically are integrating this-- for n equals 1, that's what the sine pi over L x looks like. You're integrating that and multiply it by 0. And here, you're multiplying by a constant and 0. Clearly, this is not going to give you a 0.

Let's take the second one, the second harmonic in this calculation. You're multiplying this y by this function and integrating it, OK? Notice the sine function is symmetric about 0 and so is this symmetric. Except here, this is an anti-symmetric function. So the sine is negative to the right of the middle and positive to the left.

If I multiply this positive by that, I get a positive number. If I multiply this by that, I get a negative number. The two are equal. And so the integral across there is 0. So n equals 2 is the lowest harmonic for which An is 0.

When I let that string go, the second harmonic will not be excited. Which is the lowest n for which it is excited-- in other words that An is not equal 0. Well, I told you when n equals 1, clearly An is not 0. So that will be excited. So the first harmonic will be excited.

What is the amplitude? Well, we have to calculate it. I told you, every An-- I told you what it is. I have to do some work. So you take A of 1 is 2/L. The integral from that to the H times this function where n is 1, all right?

And that, fortunately, is an easy integral to do. Integrate the sine, you get the cosine. And putting in the limits, you get this answer. So that is our answer to that part of the problem.

The final part of the problem was they said, OK, I have this shape. I let go. What will that shape look like at a time was L-- in the problem, the time was L times the square root of mu over T. T's the tension in the string. But I know that square root of mu over t is 1/d, the phase velocity. So they ask us to calculate the shape of that string at the time which is the length of the string divided by the phase velocity of progressive waves on the string.

OK, now we know completely what is the shape of the string. It is this formula. This is the formula that tells us what the string looks like at all places, all times. From our knowledge of this integral, depending where it's symmetric or anti-symmetric sinusoidal function about the center, we know that every second one will be 0.

So this sum now, I've only summed for n equals 1, 3, 5, 7, et cetera. The terms n equals 2, 4, 6, et cetera, will have 0 amplitude. But this now describes our situation in its entirety-- because I've now not only know the spatial shape, but also as a function of time. So this describes what it was at time t. What will it be if I increase the time-- let's start from 0, and I go to a time L/v?

All right, well, if this changes from 0 to L/v, this changes to cosine n pi. But you know that for n equals 1, 3, 5, et cetera, cosine of n pi-- in other words, at pi, at 3 pi, et cetera-- is minus 1. So at this later time, every term in this expansion has changed signs. But it's exactly the same series, except it's minus 1 here.

So what has happened to the string? It is exactly the same shape with an opposite sign. So with the string, instead of being up here like that like it was over here, it's flipped over like that. Without doing any more work, I can conclude that simply because I have the same series as before with a negative sign. So that is the answer.

So I've solved the problem. But at this instance, I just want to digress for a second and tell you this actually is a problem that we could have almost done all by thinking alone. It wasn't necessary to do big fraction of the work we did. The only part which there was no choice is this amplitude [INAUDIBLE].

Because I could have gone back, scratched my memory and knowledge, and said, look, hold on. When I have been a string, I could always analyze it in terms of normal modes. But also, we know that we can describe the solutions in terms of progressive waves. The two are equivalent to each other. It's not extra solutions. But by adding appropriately normal modes, I can get a superposition of progressive waves.

I can use the uniqueness theorem to see if I've got a solution which represents the situation at hand. If you look at the original problem, if you look at this problem, this is stationary, all right? So I could imagine this solution being the superposition of two progressive waves that look like this but half the amplitude, one moving to the left and one moving to the right.

As they overlap-- in other words, I have two waves, one like this moving to the right, and one like this moving to the left. If I add them, I'll get this distortion, which is stationary. Therefore, if I solved the problem for each one of these progressive waves and add them, I'll get the answer to this. It'll satisfy all the boundary conditions. By the uniqueness theorem, it will be the correct description of what happened.

And then with that, I can predict what will be the shape at any time. So for example, in the time they ask L/v, a progressive wave will move a distance L. So the wave over here, which was going to do the right would hit this boundary-- that pulse-- would flip over. And you know that this string is fixed there.

What happens if a pulse comes to a fixed end? Imagine you are causing the pulse. You are the forcewave moving along which causes the distortion. When you come to the rigid end, you're pulling. It won't give. So the only way to happen, the string will pull you down. So you'll flip over.

So this pulse, when it hits this end, will flip over upside down, and then progress this way upside down. This one will go to this end, flip over, and progress back. By the time t has changed by L over the velocity, those flipped pulses would have come back here and overlapped, but upside down. And so this would have flipped.

By this kind of analysis, if you use your wits about it, sometimes you can solve more difficult problems quickly by using all the knowledge you have about waves, the propagation, et cetera. Probably a good time to stop. Thank you.