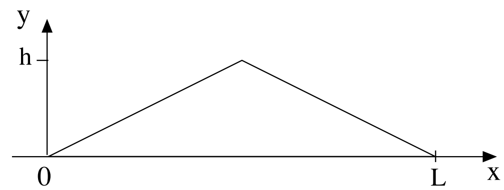
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #5 Solutions

Problem 5.1 (French 6-12)[1] – Plucked string

A sketch of the string is shown.



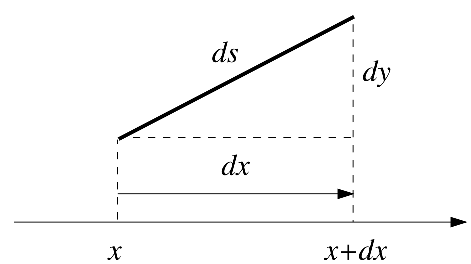
a) Remember that the kinetic energy density of a wave $y(x,t)$ in a string is $\dfrac{dK}{dx} = \dfrac{1}{2}\mu\left(\dfrac{\partial y}{\partial t}\right)^2$, and the potential energy density is $\dfrac{dU}{dx} = \dfrac{1}{2}T\left(\dfrac{\partial y}{\partial x}\right)^2$. Here μ is the mass density and T is the tension in the string.

Then, the total energy of the string at $t = 0$ is

$$\begin{aligned} E &= K + U = U \quad (K = 0 \text{ at } t = 0) = \int \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx \\ &= \frac{1}{2}T\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}TL\left(\frac{2h}{L}\right)^2 \qquad E = \frac{2h^2T}{L}. \end{aligned}$$

Since energy is conserved (we ignore any form of damping) the energy at $t = 0$ is the same as the energy at later times.

Alternatively, we can calculate the potential energy of the string directly. The potential energy can be calculated by finding the amount by which the string, when deformed, is longer that when it is straight. This extension, multiplied by the assumed constant tension T, is the work done by us creating the deformation. A displaced infinitesimal segment of a string is shown in the figure.



Thus, for the segment, we have $dU = T(ds - dx)$, where $ds = \sqrt{dx^2 + dy^2}$ $= dx\sqrt{1 + \left(\dfrac{\partial y}{\partial x}\right)^2}$. If we assume that the transverse displacements are *small*, so that $\partial y/\partial x \ll 1$, we can approximate the above expression using the binomial expansion to two terms: $ds - dx \approx \dfrac{1}{2}\left(\dfrac{\partial y}{\partial x}\right)^2 dx$. Therefore,

$$dU \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx \quad \Rightarrow \frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2.$$

b) From our choice of coordinates, the shape of the string and that of all subsequent oscillations are odd functions. Hence, we can apply a Fourier transform to decompose the motion of the wave into sine functions only. They will have the form $y_n(x,t) = A_n \sin \omega_n t$, where A_n is the amplitude of

[1] The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

the n-th harmonic, $\omega_n = n\omega_1$ and ω_1 is the angular frequency of the first harmonic (fundamental). The initial shape of the string repeats at an angular frequency of ω_1 because all harmonics repeat at an integer multiple of the first harmonic.

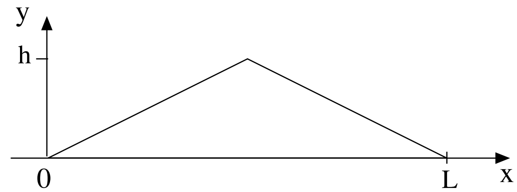
We can compute ω_1 from the relation $\omega_n = k_n v$. We know that the first harmonic has a wavelength $\lambda_1 = 2L$. Hence, $k_1 = 2\pi/\lambda_1 = \pi/L$. Therefore, $\omega_1 = \pi v/L = \pi/L\sqrt{T/\mu}$. Then, the initial pulse shape repeats every $2L\sqrt{\mu/T}$ seconds. Notice that this is the travel time of a pulse from one end of the string to the other, and back.

Problem 5.2 – Fourier analysis

a) The function is

$$y(x) = \begin{cases} \frac{2h}{L}x & \text{if } 0 \le x < L/2 \\ -\frac{2h}{L}x + 2h & \text{if } L/2 \le x \le L, \end{cases}$$

and a sketch of $y(x)$ is shown.



The most generic Fourier expansion is $y(x) = \sum_{n=0}^{\infty} A_n \cos(k_n x) + B_n \sin(k_n x)$. Since $f(0) = 0$, all cosine terms will vanish. Furthermore, $y(L) = 0 \quad \sum_{n=0}^{\infty} B_n \sin(k_n L) = 0$. Since, in general, $B_n \neq 0$ then, $\sin(k_n L) = 0 \quad \Rightarrow k_n L = n\pi \qquad k_n = \dfrac{n\pi}{L}$.

Hence, the Fourier expansion of $y(x)$ is $y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$. Notice that the sum starts at $n = 1$. The $n = 0$ term equals zero so it does not contribute. We can find the value of B_n by multiplying both sides by $\sin(m\pi x/L)$ and integrating with respect to x:

$$\begin{aligned} \int_0^L \sin\left(\frac{m\pi}{L}x\right) y(x)\,dx &= \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\,dx \\ \int_0^L \sin\left(\frac{m\pi}{L}x\right) y(x)\,dx &= \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\,dx. \end{aligned}$$

We recall the orthogonality property of the sine function,

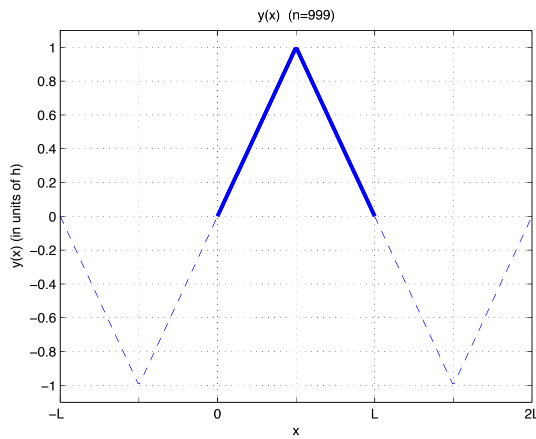
$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n. \end{cases}$$ Hence, $\int_0^L \sin\left(\frac{m\pi}{L}x\right) y(x)\,dx = B_m \dfrac{L}{2}$.

The Fourier coefficients then are

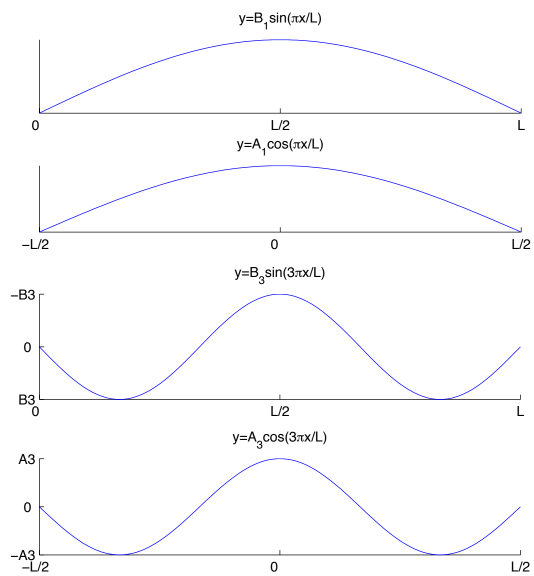
$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) y(x)\,dx = \frac{2}{L}\left[\int_0^{L/2} \sin\left(\frac{n\pi}{L}x\right)\frac{2h}{L}x\,dx + \int_{L/2}^L \sin\left(\frac{n\pi}{L}x\right)\left(-\frac{2h}{L}x + 2h\right)\,dx\right] \\ &= \frac{4h}{n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - \underbrace{\sin(n\pi)}_{=0}\right] = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

A few values of B_n are $B_1 = \frac{8h}{\pi^2} \quad B_3 = -\frac{8h}{9\pi^2} \quad B_5 = \frac{8h}{25\pi^2}$. Note that B_n is zero for all even n and that the sign of B_n alternates for odd n. We could have predicted that. Why?

The Fourier expansion of $y(x)$ then is $y(x) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}x\right)$. A graph of $y(x)$ for values $x < 0$ and $x > L$ and $n = 1 \to 999$ is shown in the figure below.



Note that the spatial period of this function is $2L$ and the mean value over this period is zero. Alternatively, we could have shifted the function so that the peak was at $x = 0$ and expanded in terms of cosines over a spatial period of 2L. All functions would then be even. Since all we are doing is shifting the function by $L/2$, we expect that the Fourier coefficients of the sine expansion, B_n, are equal in magnitude to the Fourier coefficients A_n in the cosine expansion $y(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$.



It easy to see why the magnitudes of the coefficients of the cosine and sine series must equal. Consider the graphs of the first harmonic for each series. It is then clear that $A_1 = B_1$. The case for the third harmonics is similar. Remember that $B_3 < 0$. Then, $A_3 = -B_3$. We thus have $A_1 = B_1 \quad A_3 = -B_3 \quad A_5 = B_5 \quad A_7 = -B_7 \ \ldots$ Alternatively, we could have computed the Fourier expansion where the spatial wavelength is L. In that case, the decomposition of the function (peak at $x = 0$)

$$y(x) = \begin{cases} \frac{2h}{L}x + h & \text{if } -L/2 \leq x < 0 \\ -\frac{2h}{L}x + h & \text{if } 0 \leq x \leq L/2, \end{cases}$$

would have the form

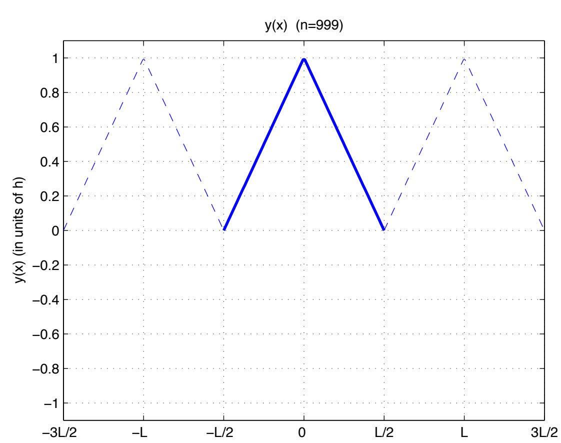
$$y(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{2n\pi}{L}x\right).$$

Convince yourself that sine terms are not allowed in this particular (even) Fourier decomposition. The Fourier expansion, in this case, would be

$$y(x) = \frac{h}{2} + \sum_{n=1}^{\infty} \frac{2h}{n^2\pi^2}\left(1 - \cos n\pi\right) \cos\left(\frac{2n\pi}{L}x\right).$$

Note that the constant term $h/2$ is the average value of $y(x)$ over one spatial period. This constant comes from the $n = 0$ term.

The graph of this Fourier expansion is shown to the right. This expansion now is even, has a nonzero mean average ($h/2$) and a spatial period L.

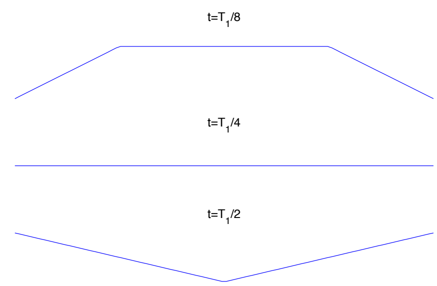


Since this Fourier decomposition gives the shape of the original function in the interval $[-L/2, L/2]$, it is a correct mathematical solution. In part (c), however, we will see that this decomposition is not physically correct if we let the string evolve in time.

b) We know how sinusoids evolve in time. For example, the sinusoid $y(x) = A\sin(kx)$ evolves as $y(x,t) = A\sin(kx)\cos(\omega t + \phi_t)$, where ω is the frequency of oscillations given by the dispersion relation and ϕ_t is the temporal phase of the oscillations. The initial condition $y(x,0) = y(x)$ requires $\phi_t = 0$. Each Fourier component of the string shape $B_n \sin(k_n x)$ will evolve as $B_n \sin(k_n x)\cos(\omega_n t)$, where $\omega_n = k_n v = n\pi\sqrt{T/\mu}/L$. The string shape then evolves as

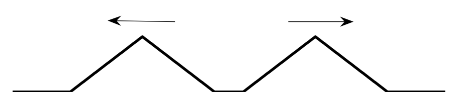
$$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi}{L}vt\right).$$

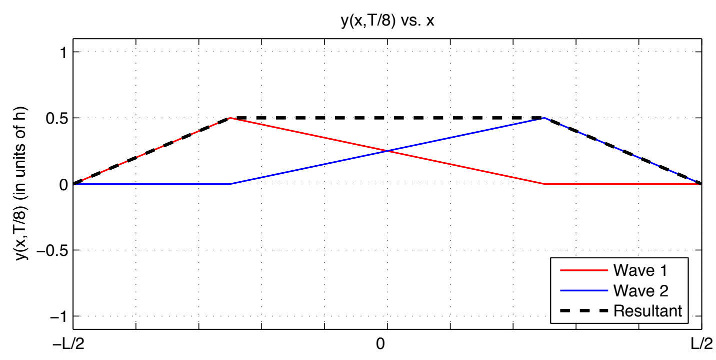
where $v = \sqrt{T/\mu}$ is the speed of propagation. Could we also have said that the shape of the string evolves as $y(x,t) = \dfrac{h}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2h}{n^2\pi^2}(1-\cos n\pi)\cos\left(\frac{2n\pi}{L}x\right)\cos\left(\frac{2n\pi}{L}vt\right)$?



The answer is NO! Try it, you will notice that at $t = T_1/4$ the entire string is at position $h/2$ (the ends are no longer fixed). Shown are the superpositions of the Fourier standing waves for $t = T_1/8$, $T_1/4$ and $T_1/2$ ($n = 1 \to 999$).

c) There is an alternative way of thinking about the time evolution. The moment you release the string, one triangle (height $h/2$) will travel to the right and the other to the left. The boundary condition $y(\pm L, t) = 0$ must hold at all times.





The graph shows the traveling waves and their sum at $t = T/8$. Recall that fixed string ends imply a reflection coefficient of -1. Hence, incident waves flip at the ends of the string. Initially, Wave 1 travels to the left and Wave 2 travels to the right. Notice that $999/2$ evolving standing waves and the 2 traveling waves give results that are indistinguishable.

Problem 5.3 Fourier series

The most generic Fourier expansion is $y(x) = \displaystyle\sum_{n=0}^{\infty} A_n \cos(k_n x + \phi_n)$. The functions in this problem have the boundary conditions $y(0,t) = y(L,t) = 0$, which imply $\phi_n = \pi/2$ and $k_n = \pi n/L$. Hence,

$$y(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right).$$

Note that the sum now starts from $n=0$ rather than $n=1$. The $n=0$ term equals zero so it does not contribute to the sum. The Fourier coefficients, A_n, can be found by multiplying the latter expression by $\sin(k_m x)$ and integrating:

$$\begin{aligned}\int_0^L \sin(k_m x)\, y(x)\, dx &= \int_0^L \sin(k_m x) \sum_{n=1}^{\infty} A_n \sin(k_n x)\; dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin(k_n x)\sin(k_m x)\; dx \\ &= A_m \frac{L}{2} \quad \Rightarrow A_m = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}\right)\; dx\end{aligned}$$

a) The function is $y(x) = Ax(1-x)$ From our discussion above,

$$\begin{aligned}A_n &= \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}x\right)\; dx = \frac{2}{L}\int_0^L Ax(1-x)\sin\left(\frac{n\pi}{L}x\right)\; dx \\ &= \frac{2AL^2}{\pi^3 n^3}\left(2 - \underbrace{2\cos n\pi}_{+1 \text{ n even -1 n odd}} - n\pi \underbrace{\sin n\pi}_{=0\,\forall n}\right) \quad \Rightarrow A_n = \frac{8AL^2}{\pi^3}\frac{1}{n^3} \quad n = 1,3,5,7\ldots\end{aligned}$$

b) The Fourier expansion of a trigonometric function is itself. By inspection, the Fourier coefficients are $A_n = \begin{cases} A & \text{if } n=1 \\ 0 & \text{if } n \neq 0. \end{cases}$ More formally,

$$A_n = \frac{2}{L}\int_0^L A\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\; dx = \begin{cases} A & \text{if } n=1 \\ 0 & \text{if } n \neq 0. \end{cases}$$

c) The function is $y(x) = \begin{cases} A\sin\left(\frac{2\pi}{L}x\right) & \text{if } 0 \leq x < L/2 \\ 0 & \text{if } L/2 \leq x \leq L. \end{cases}$ Hence, the Fourier coefficients are

$$A_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}\right)\; dx = \frac{2A}{L}\int_0^{L/2} \sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\; dx = -\frac{4A}{\pi}\frac{1}{n^2-4}\sin\left(\frac{n\pi}{2}\right)$$

We now must be careful because A_n is ill-defined at $n=2$. We can evaluate A_2 using L'Hopital's rule: $A_2 = -\frac{4A}{\pi}\left.\frac{\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right)}{2n}\right|_{n=2} = \frac{A}{2}$.

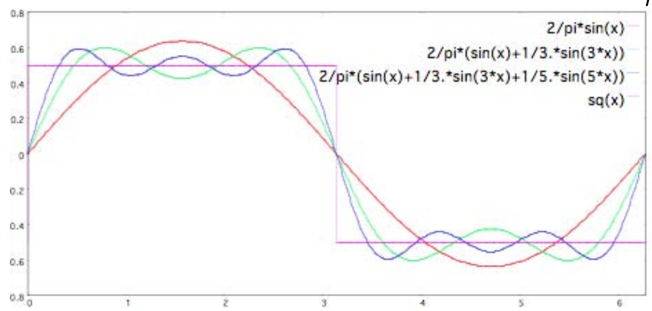
When n is even (except $n=2$), $A_n = 0$. Hence, $A_n = \begin{cases} 0 & \text{if } n = 4,6,8\ldots \\ A/2 & \text{if } n=2 \\ \frac{4A}{\pi}\frac{\sin(n\pi/2)}{4-n^2} & \text{if } n = 1,3,5,7\ldots \end{cases}$

Problem 5.4 – Fourier series for a square wave

Since the function is symmetric around 0, the A_0 term is zero, and since the square wave is odd, the other A_n terms are also zero. The B_n terms are: $B_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\left(\frac{n\pi x}{\pi}\right)\; dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\; dx$. Again, because the square wave is odd $B_n = 0$ for $n=0$ and all even values of n. For the odd values of n, we can simplify the integral since we only need to integrate over the first half of the wave and multiply by 2: $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$. But the amplitude of 0.5 in $f(x)$ cancels this factor of two and the final result is $B_n = \frac{2}{n\pi}$.

$f(x) = \frac{2}{\pi}\left[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right]$

The figure shows the first, first+second, and first+second+third terms as red, blue, and green lines, respectively. Notice how each added term partly "corrects" the places where the previous sum misses the desired function.



Problem 5.5 – Phase and group velocity in your bathtub

The dispersion relation for deep-water waves is approximately $\omega^2 = gk + \frac{T}{\rho}k^3$, where $\omega = 2\pi/\lambda$.

a) For very short wavelengths ($\lambda \ll 1.7$ cm), the k^3 term dominates. Then $\omega^2 \approx T/\rho k^3$. Then, the phase velocity is $v_p = \frac{\omega}{k} = \sqrt{\frac{Tk}{\rho}}$. The group velocity is $v_g = \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{Tk}{\rho}}$. Combining these two equations gives $v_g = 3/2 v_p$.

b) For very long wavelengths ($\lambda \gg 1.7$ cm), the k term dominates. Then $\omega^2 \approx gk$. Then, the phase velocity is $v_p = \sqrt{\frac{g}{k}}$ and the group velocity is $v_g = \frac{1}{2}\sqrt{\frac{g}{k}}$. Hence, $v_g = v_p/2$.

Problem 5.6 – Shallow-water waves (Home experiment)

This experiment was performed by Igor Sylvester, an 8.03 students in 2004.

a) I made many measurements and finally concluded that it took about 3 s for a wave packet to travel 4 times the diameter (23 cm) of a pan with a depth of 9 mm. The uncertainty in this is about 0.5 s (17%). I used a stopwatch that can measure time with an accuracy of 10 ms, but the uncertainty is much larger because it's not easy to tell precisely where the packet is.

b) The speed of the wave packet based on my results is 31 ± 5 cm/s. This is in good agreement with the predicted value of 29.7 cm/s.

Problem 5.7 – (French 7–20) Why are deep-water waves dispersive?

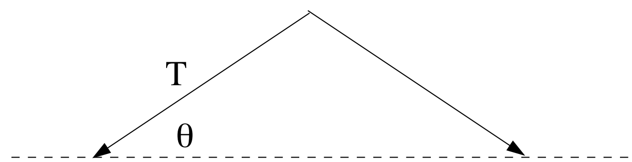
a) The potential energy of the liquid is $U = mgh = (\rho A y)gy = \rho A g y^2$. The kinetic energy is $K = \frac{1}{2}mv^2 = \frac{1}{2}(\rho A l)\left(\frac{dy}{dt}\right)^2$. Then, we can derive the equation of motion from conservation of energy: $\frac{\partial E}{\partial t} = 0 = (2A\rho g y + \rho A l \ddot{y})\dot{y} \;\Rightarrow \ddot{y} + \frac{2g}{l}y = 0$. This is a simple harmonic oscillator. Hence, the period of oscillations is $T = \pi\sqrt{2l/g}$.

b) We know that $v = \nu\lambda$. Assuming that $\lambda \approx 2l$, $v = 2\nu l = \sqrt{g\lambda}/\pi$.

c) For $\lambda = 500$ m, $v = 27$ m/s ≈ 97 km/h ≈ 61 mi/h.

Problem 5.8 – Energy in waves

a) Using $v = \sqrt{T/\mu}$ and $\nu = v\lambda$, Eq. (7-38) on page 242 in French, which gives the energy per wavelength in a traveling wave, can be rewritten: $W_{\text{cycle}} = 2\pi^2\nu^2A^2\lambda\mu = 2\pi^2A^2\frac{T}{\lambda}$. The equation $E_\lambda = \frac{\pi^2A^2T}{\lambda}$ is the energy stored in one wavelength of a standing wave. Note that $W_{\text{cycle}} = 2E_\lambda$. This is correct because the energy per wavelength in a traveling wave is double that of a standing wave (same amplitude).



b) The graph shows the deformed string (highly exaggerated). If the tension remains approximately constant (for modest distortion) then the work needed to pick up the string is
$W = \int_0^A F(y)\,dy$, where

$F(y) = 2T\sin\theta \approx 2T\frac{y}{L/2} = \frac{4T}{L}y$. Then, $W = \int_0^A \left(\frac{4T}{L}\right) y\,dy = \frac{2TA^2}{L}$.

c) $W_{\text{TOT}} = nW = n\int_0^{A_n}\left(\frac{2T}{L/2n}\right) y\,dy = \frac{2Tn^2A_n^2}{L}$.

d) For the triangular wave, $L = n\lambda/2$ and $E_{\lambda_{\text{triangle}}} = \frac{2}{n}\frac{2Tn^2A_n^2}{n\lambda/2} = \frac{8TA_n^2}{\lambda}$.

Then, the energy ratio is $\frac{E_{\lambda_{\text{sine}}}}{E_{\lambda_{\text{triangle}}}} = \frac{\pi^2}{8} \approx 1.25$.

Problem 5.9 – Energy in traveling waves on a string

a) A standing wave with amplitude A can be created by two traveling waves, moving in opposite directions, each with amplitude $0.5A$. Thus, the total energy (per wavelength λ) is half that of the standing wave with amplitude A. When the standing wave stands still, all it energy is in the form of potential energy, which is proportional to A^2. For one of the two traveling waves (amplitude $0.5A$), the potential energy is proportional to $A^2/4$ and it is independent of time. Thus, its kinetic energy (at any moment in time) must also be $A^2/4$, so that its total energy per wavelength is half that of the standing wave.

b) Let's calculate the kinetic and potential energies in one wavelength explicitly. The wave is $y(x,t) = A\sin(\omega t - kx)$, where $k = 2\pi/\lambda$, $\omega = vk$ and $v^2 = T/\mu$. The kinetic energy is

$$K = \int_0^\lambda \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{\mu}{2}\int_0^\lambda A^2\omega^2\cos^2(\omega t - kx)\,dx = \frac{\mu A^2\omega^2}{2}\frac{\lambda}{2} = \frac{TA^2\pi^2}{\lambda}.$$

The potential energy is

$$U = \int_0^\lambda \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{T}{2}\int_0^\lambda A^2k^2\cos^2(\omega t - kx)\,dx = \frac{TA^2k^2}{2}\frac{\lambda}{2} = \frac{TA^2\pi^2}{\lambda}.$$

As expected, the kinetic and potential energies are equal. The total energy in one wavelength of a traveling wave is $2TA^2\pi^2/\lambda$.