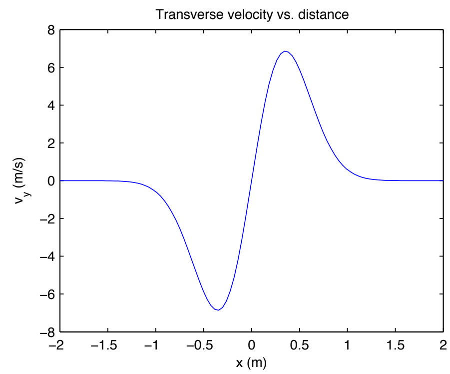
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

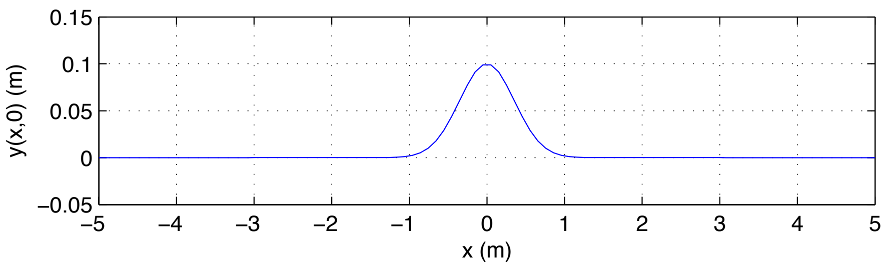
Problem Set #4 Solutions

Problem 4.1 (French 7-12)[1] – Traveling pulse



a) Since the pulse is traveling to the right, the piece of string on the right side of the peak is "rising" and the piece on the left is "falling." The transverse velocity of the peak is zero but it has the maximum acceleration (see the figure).

b) The pulse shape is shown below. We can model the pulse with a Gaussian function. That is, the pulse resembles $y(\eta) = Ae^{-\alpha\eta^2}$ where $\eta = x - vt$, $A = 0.1$ m and $\alpha = 4$ m^{-2}. The graph of the pulse shape is actually this function.



The transverse velocity is then $\dfrac{\partial y}{\partial t} = -2A\alpha\eta e^{-\alpha\eta^2}\dfrac{\partial \eta}{\partial t} = 2A\alpha v\eta e^{-\alpha\eta^2}$ We can find the maximum transverse velocity at $t = 0$ by requiring that

$$0 = \left.\frac{\partial^2 y}{\partial t^2}\right|_{t=0} = 2A\alpha v^2 e^{-\alpha\eta^2}\left(2\alpha x_{\max}^2 - 1\right) \Rightarrow 0 = 2\alpha x_{\max}^2 - 1 \Rightarrow x_{\max} = \sqrt{\frac{1}{2\alpha}}$$

Hence, the maximum transverse velocity at $t = 0$ is

$$\left.v_y\right|_{\max} = \left.\frac{\partial y}{\partial t}\right|_{x=x_{\max}} = 2Av\sqrt{\frac{\alpha}{2}}e^{-1/2} \approx 6.86 \text{ m/s}$$

c) The mass density of the string is $\mu = 1/50$ kg/m. The tension in the string is $T = \mu v^2 \approx 32$ N.

d) Any wave traveling in the negative x direction with a speed v can be described as $y(x,t) = f(\eta) = f(kx + \omega t)$ where $f(\eta)$ is the shape of the wave, k is the wave number and ω is the angular frequency. For sinusoidal waves: $y(x,t) = A\sin(kx + \omega t + \phi)$, where A is the amplitude of the wave and ϕ is the phase of the sinusoid. Furthermore, a wavelength of 5 m implies $k = 2\pi/\lambda = 0.4\pi$ m^{-1}. Since this wave is traveling on a string, it must obey the relation $\omega = kv = 16\pi$ s^{-1}. Therefore, the equation describing the wave is

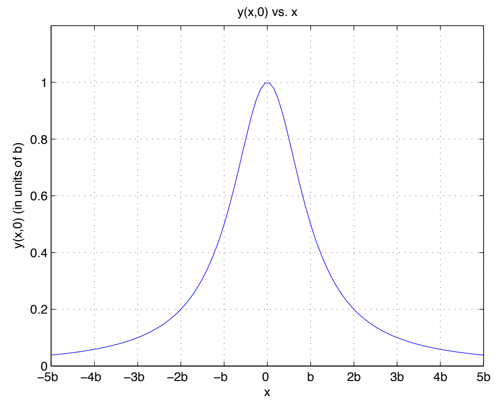
$$y(x,t) = (0.2 \text{ m})\sin\left(\left(0.4\pi \text{ m}^{-1}\right)x + \left(16\pi \text{ s}^{-1}\right)t + \phi\right),$$

[1] The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

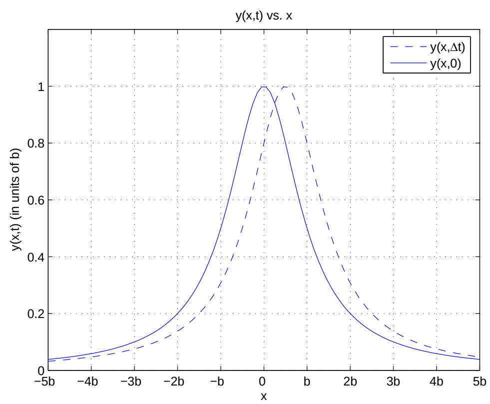
where ϕ is unknown since the phase of the wave was unspecified.

Problem 4.2 (French 7-13) – Traveling pulse

a) A sketch of $y(x, 0)$ is shown.



b) Remember that any pulse or wave traveling in the positive x-direction can be expressed as $y(\omega t - kx)$, for $k \geq 0$ and that its speed of propagation is $v = \omega/k$. Then, letting $z = \omega t - kx$ and expressing $y(x, t)$ as a function of z, $y(z) = \dfrac{b^3}{b^2 + z^2}$. Hence, $z = 2x - ut$. Therefore, for positive values of u, the pulse travels in the positive x direction with a speed $v = u/2$.



c)

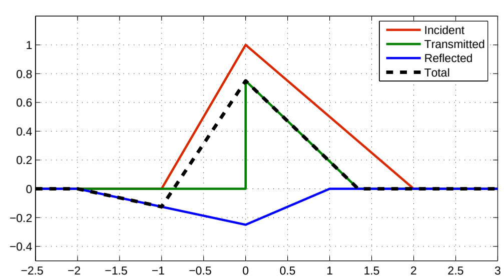
$$\begin{aligned} v_y(t=0) &= \left.\frac{\partial y}{\partial t}\right|_{t=0} \\ &= \left.\frac{2b^3 u(2x - ut)}{\left(b^2 + (2x - ut)^2\right)^2}\right|_{t=0} \\ &= \frac{4b^3 xu}{\left(\ b^2 + 4x^2\right)^2} \end{aligned}$$

Problem 4.3 – Pulse reflection at a boundary

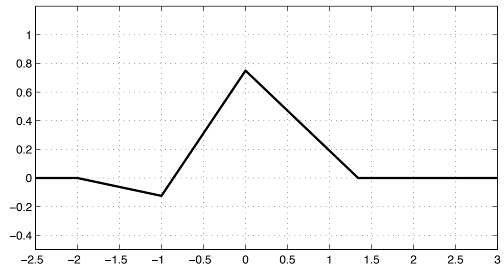
a) The propagation speed in string 1 is $v_1 = \sqrt{T/\mu_1} = 10\sqrt{2}$ m/s ≈ 14 m/s and in string 2, $v_2 = \sqrt{T/\mu_2} = 10\sqrt{2/3}$ m/s ≈ 8 m/s. Then, the reflection and transmission coefficients are

$$R = \frac{v_2 - v_1}{v_1 + v_2} = \frac{\sqrt{3} - 3}{\sqrt{3} + 3} \approx -\frac{1}{4} \qquad T = \frac{2v_2}{v_1 + v_2} = \frac{2\sqrt{3}}{\sqrt{3} + 3} \approx \frac{3}{4}$$

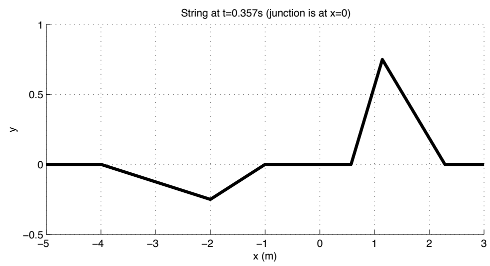
b) This graph shows the incident, reflected and transmitted waves when the pulse peak arrives at the junction ($x = 0$). Note that the reflected pulse is upside down and flipped right to left. Also, the transmitted pulse is narrower. Keep in mind that only the dashed black line is physical. The other lines (in red, green and blue) are there only for illustrative purposes.



This graph shows the total deformation of the string when the peak is at $x = 0$.



c) The shape of the string at time: $t = (5\text{ m})/v_1 = 0.357$ s is shown in this figure.



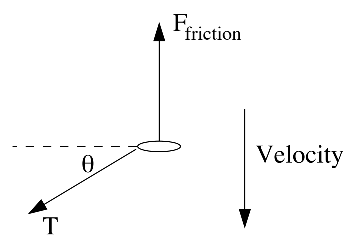
d) The sharp cusps of the pulse are unphysical because it leads to an infinite potential energy of the string. Recall that the potential energy density of a string is $\dfrac{dU}{dx} = \dfrac{1}{2}T\dfrac{\partial y}{\partial x}$. Since the pulse is not smooth at the cusp the slope is infinite. Therefore, the potential energy of the string is infinite.

Alternatively, we could argue that, at any point in the string, the forces must cancel because each point has an infinitesimally small mass. We need vanishing forces in the presence of a vanishing mass so the acceleration remains finite. The cusps in the string cause an infinite acceleration since the forces at those points do not cancel.

Problem 4.4 – Boundary conditions on a string

a) The sketch on the right shows the forces acting on the hoop. Applying Newton's second law gives $F = ma = \Delta m\ddot{y} = -T\sin\theta + F_{friction}$. Assuming that oscillations are small, $\Delta m\ddot{y} = -T\dfrac{\partial y}{\partial x} - b\dfrac{\partial y}{\partial t}$. Since the mass of the hoop is negligible, $-T\dfrac{\partial y}{\partial x} - b\dfrac{\partial y}{\partial t} = 0$

$\Rightarrow\ \dfrac{\partial y}{\partial x} = -\dfrac{b}{T}\dfrac{\partial y}{\partial t}$ at the hoop for all times.



b) Let's take the superposition of an incident wave and a reflected wave

$$y(x,t) = \underbrace{f(x - vt)}_{\text{Incident, known.}} + \underbrace{g(x + vt)}_{\text{Reflected, unknown.}} .$$

We now use the boundary condition at the hoop to solve for $g(x + vt)$. The respective derivatives are $\dfrac{\partial y}{\partial x} = f'(x - vt) + g'(x + vt)$ and $\dfrac{\partial y}{\partial t} = v\left(-f'(x - vt) + g'(x + vt)\right)$. If the hoop is at $x = 0$, then $f'(-vt) + g'(vt) = \dfrac{bv}{T}\left(f'(-vt) - g'(+vt)\right)$ so $g'(vt) = \dfrac{bv/T - 1}{bv/T + 1}f'(-vt)$.

Letting $\eta = vt$ and integrating with respect to η,

$$\int g'(\eta)\,d\eta = \int \frac{bv/T-1}{bv/T+1} f'(-\eta)\,d\eta \qquad g(\eta) = \frac{bv-T}{bv+T}(-1)f(-\eta)$$

$$g(\eta) = \frac{T-bv}{T+bv} f(-\eta).$$

Note that the integration constant must be zero for the limiting cases discussed in part (c) to hold.

c) For $b = 0$, the hoop behaves as a free end. Our result gives $g(\eta) = f(-\eta)$, which is correct since the wave is reflected without flipping. For $b \to \infty$, the hoop behaves as a clamped end. Our result gives $g(\eta) = -f(-\eta)$, which is correct since the wave is reflected flipped over. Note that for the special case when $b = T/v$, $g(\eta) = 0$. Hence, there is no reflected wave. This is known as a matched load.

Problem 4.5 – Boundary conditions in a pipe

The wave equation for the over-pressure $p(z,t)$ inside a pipe is $\frac{\partial^2 p}{\partial z^2} = \frac{\rho_0}{\kappa}\frac{\partial^2 p}{\partial t^2}$. The solution to this equation is $p(z,t) = [A\cos kz + B\sin kz]\cos\omega t$. Since the pipe is open at both ends (remember, p is over-pressure), $0 = p(0,t) = A\cos\omega t \Rightarrow A = 0$ and $0 = p(L,t) = B\sin kz\cos\omega t \Rightarrow \sin kz = 0$ $\Rightarrow k = \frac{n\pi}{L}$ where $n = 1,2,3\ldots$ We can obtain the dispersion relation by inserting $p(z,t)$ into the wave equation for the system. The relevant derivatives are

$$\frac{\partial p}{\partial z} = kB\cos kz\cos\omega t \qquad \frac{\partial p}{\partial t} = -\omega B\sin kz\sin\omega t$$

$$\frac{\partial^2 p}{\partial z^2} = -k^2 B\sin kz\cos\omega t \qquad \frac{\partial^2 p}{\partial t^2} = -\omega^2 B\sin kz\cos\omega t$$

The wave equation then reduces to

$$-k^2 B\sin kz\cos\omega t = -\omega^2 B\sin kz\cos\omega t \quad\Rightarrow\quad \omega = \sqrt{\frac{\kappa}{\rho_0}}k \quad\Rightarrow\quad \omega_n = n\frac{\pi}{L}\sqrt{\frac{\kappa}{\rho_0}}$$

Finally, the initial condition determines k_n and B. The initial condition is

$$p_0 = p(L/2,0) = B\sin k\frac{L}{2} = B\sin\frac{n\pi}{2} \quad\Rightarrow\quad B = \pm p_0 \qquad \text{if } n = 1,3,5,7\ldots$$

Hence, n must be an odd integer. Otherwise, B would equal zero and $p(z,t) = 0$ which is indeed a *trivial* solution. Finally, the wave number is $k_n = \frac{n\pi}{L}$ where $n = 1,3,5,7\ldots$ where $B = +p_0$ for $n = 1,5,9\ldots$ and $B = -p_0$ for $n = 3,7,11\ldots$

Problem 4.6 – Normal modes of discrete vs. continuous systems

a) The most general solution for a standing wave in a string is:
$y(x,t) = A\cos(kx+\phi_x)\cos(wt+\phi_t)$. The two boundary conditions are:
$0 = y(0,t) = A\cos\phi_x \Rightarrow \phi_x = \frac{\pi}{2}$ and $0 = y(L,t) = A\sin kL \Rightarrow kL = n\pi$.
Hence, the n-th normal mode of the string is:

$$y_n(x,t) = A_n\sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t+\phi_t)\,,\ \text{where } \omega_n = n\omega_1 = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} = n\pi\sqrt{\frac{T}{ML}}$$

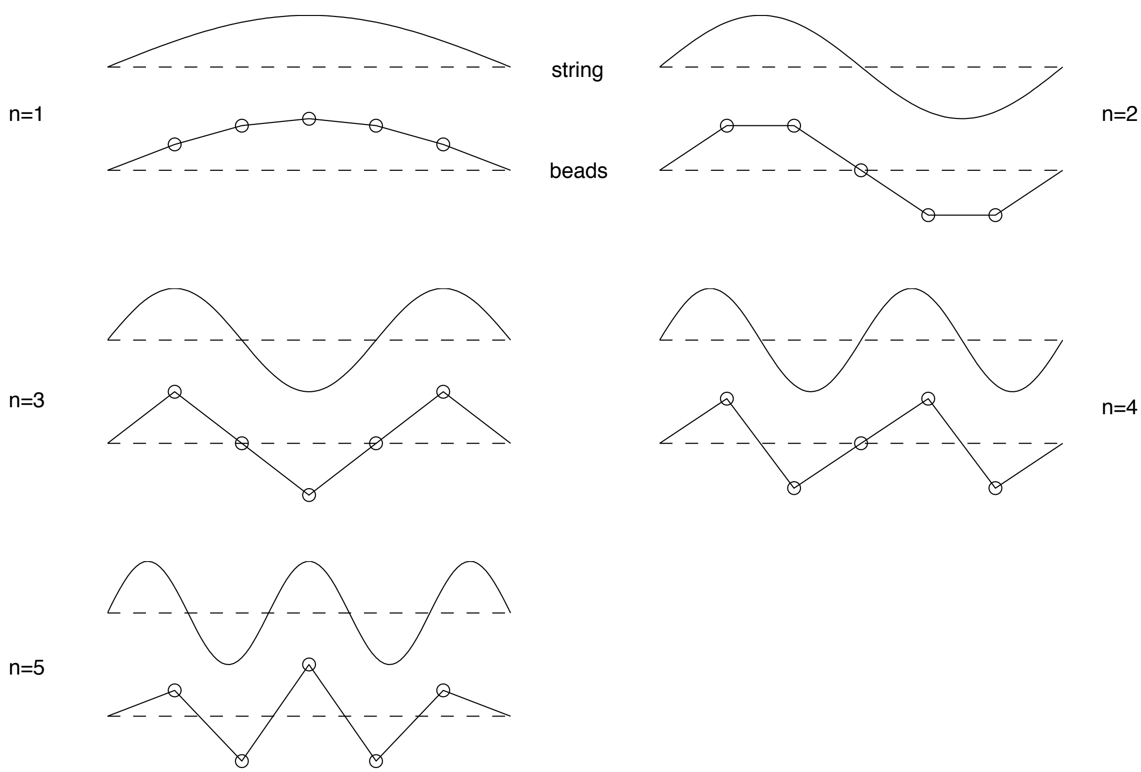
b) The general formula for the frequency of the n-th mode is $\nu_n = \frac{\omega_n}{2\pi} = \frac{n}{2}\sqrt{\frac{T}{ML}}$. The five lowest normal modes are

$$\begin{aligned}
\nu_1 &= \tfrac{1}{2}\sqrt{\tfrac{T}{ML}} \equiv \nu_0, & \nu_2 &= \sqrt{\tfrac{T}{ML}} = 2\nu_0, \\
\nu_3 &= \tfrac{3}{2}\sqrt{\tfrac{T}{ML}} = 3\nu_0, & \nu_4 &= 2\sqrt{\tfrac{T}{ML}} = 4\nu_0, \\
\nu_5 &= \tfrac{5}{2}\sqrt{\tfrac{T}{ML}} = 5\nu_0.
\end{aligned}$$

c) Using Eq. (5-25) on page 141 of French $\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \Rightarrow \nu_n = \frac{\omega_0}{\pi}\sin\left(\frac{n\pi}{2(N+1)}\right)$. The fundamental frequency is $\omega_0 = \sqrt{\frac{T}{\frac{M}{5}\frac{L}{6}}} = \sqrt{\frac{30T}{ML}} = \sqrt{120}\nu_0$. The first five frequencies are then $(N = 5)$

$$\begin{aligned}
\nu_1 &= \tfrac{\sqrt{120}}{\pi}\sin\left(\tfrac{\pi}{12}\nu_0\right) = 0.9\nu_0, & \nu_2 &= \tfrac{\sqrt{120}}{\pi}\sin\left(\tfrac{\pi}{6}\nu_0\right) = 1.7\nu_0, \\
\nu_3 &= \tfrac{\sqrt{120}}{\pi}\sin\left(\tfrac{\pi}{4}\nu_0\right) = 2.5\nu_0, & \nu_4 &= \tfrac{\sqrt{120}}{\pi}\sin\left(\tfrac{\pi}{3}\nu_0\right) = 3.0\nu_0, \\
\nu_5 &= \tfrac{\sqrt{120}}{\pi}\sin\left(\tfrac{5\pi}{12}\nu_0\right) = 3.5\nu_0.
\end{aligned}$$

d) The following figures show the first 5 normal modes for the string and the beads.



e) Since $N = 5$ is still not $N \gg 1$, the normal mode frequencies and shapes are not identical.

Problem 4.7 — Piano galore

a) The frequency of the n-th mode of a string is $\nu_n = \omega_n/2\pi = n\sqrt{T}/2L\sqrt{\mu}$. Differentiating with respect to T gives $\frac{d\nu_n}{dT} = \frac{n}{4L}\sqrt{\frac{1}{T\mu}} = \left(\frac{n}{2L}\sqrt{\frac{T}{\mu}}\right)\frac{1}{2T} = \frac{1}{2T}\nu_n$. We know that $n = 1$, $T = 250$ N, $\nu_{C_5} = 512$ Hz and $d\nu = 0.5$ Hz, so $dT = (0.5)(500)/512$ N ≈ 0.5 N.

b) Pianos have 88 keys. Many notes have two strings and many have three; some have only one string. A Steinway grand piano has a total of 216 strings. This translates into $F = 216 \times 250\ N \approx 5.4 \times 10^4$ N. This is huge; it's about the weight of a mass of 54 thousand kg (54 tons)!!

c) The G_5 will excite the second harmonic of C_4 and you will hear G_5. The fundamental of G_5 will not excite G_6. However, the second harmonic of G_5 will excite G_6 and you will hear G_6.

d) A note which is a higher harmonic of G_5 will be excited (eg. G_6, D_7, G_7, B_7). Also a note below G_5 which has G_5 as one of its higher harmonics will be excited (e.g. $G_4, C_4, G_3, E_3^b, C_3$, etc.).

e) No string is perfectly flexible and perfectly continuous. Furthermore, the restoring force on the string is linear only to a first approximation, so it is not possible for the strings to possess harmonics in perfect multiples of each other. Very shortly we will learn that the velocity is a function of frequency (or λ); a phenomenon called dispersion. So far, we always assumed ideal strings for which $v = \sqrt{T/\mu}$ (independent of ν).

There is another reason for the difference in tone between G_5 and the 6^{th} harmonic of C_3: a piano which is "in tune" is not tuned according to our scientific scale. The octaves are tuned in perfect multiples of 2 (frequency) but all other intervals are slightly altered. The perfect fifth is not so perfect after all. For more information see Waves (Berkeley Physics Course Vol. 3), by Crawford, problem 2.6 pp 91-93.

f) They had better go away since the beats are the result of a superposition of sinusoidals of the two nodes.

Problem 4.8 – Holes in woodwind instruments

a) With holes C and B closed, the pipe is 37 cm long, open at both ends. Therefore,

$$\lambda = 2L = 74\,\text{cm} \;\Rightarrow\; \nu = \frac{v}{\lambda} = 446\,\text{Hz}$$

b) If the holes are large enough this is a pipe of length 18.5 cm, open at both ends, so $\nu = 892$ Hz.

c) With only hole B closed, the effective length of the pipe is AC so $\lambda = 2(27.7\text{ cm}) = 55.4$ cm. Hence, $\nu = 600$ Hz.

d) With neither B or C closed, L is now approximately 18 cm, thus $\lambda = 2(18.5\text{ cm}) = 37$ cm. Hence, $\nu = 892$ Hz.

Problem 4.9 — Pianos can talk back

a) The sounds that you make are a superposition of different frequencies. Each string inside the piano will respond to its harmonics. Hence, the sound of your voice will be broken down into frequencies and selected frequencies will be played back by the piano. In this way, the piano is performing a Fourier analysis of your sound. The piano need not be in tune, it needs only to possess enough components to make your sound recognizable.

b) The ratios of the harmonic frequencies of the strings will not be exactly $1 : 2 : 3 \ldots$ because the piano is not tuned that way (see problem 4.7 above). In addition, the oscillations will not be in phase because of the difference in travel times of your sound to the strings (about 1 meter in 3 msec). In 3 msec the 330 Hz string will perform one complete oscillation; the 1000 Hz will make 3 oscillations, etc.

c) Apparently, phase in unimportant.

d) We cannot explain this. But it is the way our brains work. Perhaps evolution did not discover any survival value in keeping the phase.