PROFESSOR: Welcome back, and today I'm going to do for you three problems all related to driven harmonic oscillators. Now one of the things you'll notice as I'm doing these problems, that it seems I'm doing the same thing over and over again. You may have also noticed that in the other problems I did earlier for you in harmonic oscillators, et cetera.

So at this stage, I have to remind you about something. This is no accident that it is like that. It's to do with the scientific method, and I would want to briefly remind you about how we are solving problems.

Here is a diagram of a generic problem. What does it consists of? A typical problem is we are told of a physical situation in words, ordinary language. And in ordinary language, we are asked questions about given that physical situation, what will happen as a function of time? What kind of emotion it has, questions of that kind.

What problem solving is, doing the scientific method, is the following. The first thing we have to do is, by studying this situation and using the laws of nature, describing the physical situation in terms of mathematical equations. This is always possible. So starting with as I say this description, using laws of nature, we end up with a description in terms of mathematical equations.

The next thing we do is, we now forget that this has anything to do with physics. We have a set of mathematical equations, which we have to solve. This is mathematics. We use mathematics to solve this problem. All right? Once we have found the solution to this mathematical equation, there's only one thing left, we identify the answer with the questions posed in the problem.

So understanding of physics comes in this step, which I call step 1, and in the last one. The middle one, step 2, is pure mathematics. Now, it so happens that if you take all driven harmonic oscillators, the mathematics that you end up with is identical. In other words, if I look at the description of the problem in mathematical terms, I cannot tell you which problem that description belongs. That's a generic description of many, many, many situations.

Today I'll take three of them, but all three situations, all three problems-- although to you, they may look completely different-- after I've translated them in terms of mathematical equations, are identical. And so it's not surprising that from here on when we then try to solve them, et cetera, we are doing the same problem, or at least it appears so.

Having said that, let's immediately dive in and take one of the problems. So the first problem I'm going to consider is an idealized-driven RLC circuit. Why do I see idealized? Because as I've mentioned before, in real life situations, things extremely complicated, and if you take everything into account, it will take you essentially forever to get an exact solution. But we're often not most of the time not interested in the very exact one. So we simplify a physical situation to the bear essential parts and solve that problem.

So this is a problem where you have a circuit consisting all a capacitance, inductance, and a resistance, in series, which is driven by an alternating voltage source V0 cosine omega t. Before time t equals 0, I'm assuming the circuit is open. In other words, there is no charges anywhere, no current flowing.

At t equals 0, we close the circuit so the charge on the capacitor t equals 0, is 0. The current flowing in the circuit is 0. All right? And the question we want to answer is, what will the current be in the circuit at some time t a specific one, in 17 seconds or whatever, some specific time.

Since this is completely specified, I can predict what will happen. That's what it means in this case solving this problem. So let's do it. So we are now entering step 1. We're going to take the physical situation and convert it to a description in terms of mathematics. This is the conceptually hardest step of any problem. It is the one where you need to understand the physics.

It may not necessarily be the hardest or the longest. Often the solution of the equations are incredibly tedious and long and take all your time. But they are routine mathematics. Here you've got to conceptualize what's going on and translate it as I say to the language of mathematics.

So in the case of I've had have to define coordinates for this problem. So what I have is, I'll assume that at some instant of time that in this circuit, it's closed because it's past equal to 0, there is a current I flowing that on this capacitor on the left there is plus charge Q and minus charge Q on the right side that is this inductance. And over here, we have an alternating potential applied.

Now I have to be very specific at any instant of time which side is positive and negative. I'll assume that t equals 0. It's 0 over here, and at this point here, it's plus V0 cosine omega t. So these are the coordinates of this problem. And now what drives the current? What decides what happens here? What are the law of natures applicable? In this case, are the Maxwell's equations or to be specific, the Faraday equation.

All right? One more thing I should add that I is a current here flowing charge by conservation of charge. It's law of physics. The current at the instant of time, will be equal to the rate of change of the charge. So at the instant time, I is, in my language, Q with a dot on top of it.

This is plus because of the way in which I define my coordinates. Beware if you by mistake say suppose I define my coordinates this being minus and this plus, then this equation would be a minus here, conservation of charge. But with the way I've defined these coordinates, this is plus. So that's one.

Next I said what the law of nature which governs the motion of the charges is the Faraday's law, which states that if I take the line integral of the electric field around a closed loop, that's this, it equals to minus the rate of change of total magnetic flux linking this circuit, going through this circuit.

Now, you know that the magnetic flux through a circuit is equal to the current times the self-inductance of that circuit or the total inductance in that circuit. So taking this law of nature, this law of nature, I can now translate this into mathematics, and that's what I'm going to do.

So first I'm going to take the line integral of E dl around this circuit. If I go around from here to here, this is positive relative to this point, so the electric field from here to here will be in this direction, and so the line integral of E dl along here will be minus, and by definition of a voltage, it will be just minus the voltage drop across here, which is V0 cosine omega t.

So this is minus V0 cosine omega t. I continue going around. From here to here, the electric field is in this direction. And so that will be the integral V dl is Q over C, that's from the definition of a capacitance, of course. Then as we go around in these idealized diagrams, all wires have 0 resistance and therefore there cannot be field in them. That's why there's no contribution to this from the field inside the wires.

I continue. This inductance is just coiled wire, so there's still no contribution to this in the L. I continue. Here I know there is a current flowing like this means the reason the electric field in this direction and once again knowing the definition of our Ohms law, I get that the integral of the electric fields from here to here is I times R.

So this is this quantity, the integral of the electric field around this closed loop. That must equal by Faraday's law to minus the rate of change of magnetic flux, which is minus L dI dt. We're almost finished. Now we do algebra. We're playing around. I can rewrite this like this knowing that the current is the rate of change of Q, and the rate of change of current is the second derivative. So this is just straightforward algebra.

Now, I'm going to rewrite just moving terms around to make it look in the way I prefer. And so I've rewritten this equation now in this form. And now, I will just redefine some constants so that I recognize the equation better. So I will define R over L by a constant gamma. And I will define 1 over LC by the constant omega 0 squared. And I define V0 L by the constant f.

And so this equation rewritten looks like that. Eureka. This is an equation I've seen millions of times, and I know how to solve it. So what we've done is, we've taken this physical situation, translated it into a mathematical equation. This is the equation of motion, this just defines the constants, and this is the boundary conditions. It tells us that in our particular problem, we switch that circuit at t equals 0.

At that time, what was the condition of the current and charge? So Q at 0 we said was 0. Q dot at 0 was 0. This completely, mathematically defines that problem. This is the mathematical description of this problem. From now on, I don't have to even remember that this has anything to do with physics. This is now a problem in mathematics.

What is the solution of this equation satisfying these boundary conditions? We'll do that later. I'm now going to immediately go to the next problem. See in each case, I've done the physics. This is the end of the physics in essence. From now on, I just have to do mathematics. So let me take the next problem, a different one.

It seems completely different. Here we were dealing with charges, currents, circuits, et cetera, and now I'm going to look at the physical one. I could even bring you a model of that problem. What I'm going to consider here is an ideal pendulum, simple pendulum, ideal, which is a heavy mass, considered to be a point mass, a string, which is considered to be always taught but yet massless. So it's an ideal pendulum.

What I have here is an approximation to an ideal one, and if I understand what the ideal one does, this in reality, will you do something pretty close to it. And so what I'm going to consider, I am going to be oscillating my hand backwards and forwards sinusoidally starting this at some definite time, and I'll try to predict the motion of this.

And notice I can make all sorts of motion out of this. If I do it slowly, look, my hand and the ball are going in the same direction. If I go fast, they're going in opposite direction. And the way this is oscillating is different, depends on how I am moving my hand backward and forward.

So let me tell you very precisely what is this idealized situation which I am trying to solve and predict what will happen. So I have this ideal pendulum. It has a length, L, mass, m. It's in the vertical plane. Gravity is down there. At one end of the string, I will be moving my hand in perfect sinusoidal fashion. It will have an amplitude of x0 and angular frequency omega, so it's x0 sine omega t.

I am making the following assumptions, that's why it's idealized. I'm assuming it's a massless string. I'm assuming it's always taught. I'm going to assume that it's all making small oscillations so that if in any calculation, I can take a sine to be an angle to be equal to the angle.

Furthermore I'm going to assume that there is a bit of drag, there is friction, and I will assume that the frictional force on this that mass is proportional to the velocity, and the constant of that velocity is B. Now you ask me, why choose that? Well, the simple answer is, I want to find this situation when I know I can solve it. And I know how to solve this problem if it's proportional to velocity.

If it's something more complicated, then the equation may be correct at the end, but I will not be able to just like that on the board, solve it for you. I could do it numerically, use a computer, et cetera, but at the same time, I know that if I have something which is very close to being what the actual situation is, the prediction of the real world compared to this idealized one will not be very different. Qualitatively, I will get a good understanding of what's going on.

So this is the assumption, and what is the question? What do I want to predict? I want to now take a very specific situation. I want to take a situation that at t equals 0 initially, the mass is stationary, and it's exactly below my hand, but my hand is not stationary, it's moving.

So I am moving this backwards and forwards sinusoidally, all right? And in the instant, when my hand is over the ball, I let go of the ball and then I continue moving it.

So I have specified this exactly what this ball is doing at t equals 0, what my hand is doing, and I want to predict what will happen at some arbitrary later time. So my question is what will happen, and this time to make it a little more interesting also, I want to know in qualitative terms what kind of motion will it have if I move this slowly and quickly and check against reality at the end.

And [? originally ?] I could tell you that things are different when I'm doing it slowly. As I told you before, if I do this slowly like this, my hand and the ball going in the same direction. If I go fast, I can make them go in opposite directions. All that should come out of this. So the part of the question is to qualitatively understand this phenomenon. So that's my problem number 2.

The third problem that I'm going to do. Sorry. I haven't done it. That's what I want to do, and I have to do it. I wanted to get the exact things over quicker, but no such luck. Now, I'm going to solve this problem. So back again. Look at that diagram and what's the first thing to do. Step 1 use the laws of nature to restate this problem in terms of math and language of mathematics. That's what we have to do.

All right. Let's go a little fast. So I have to define some coordinate system so I redraw my picture and now I'm going to tell you that I'll take some vertical stationary axis here, and I'll say it instant t, this mass is at some position, y. My hand holding here is at position is at x0 sine omega t, time t. Fine.

That this string is taught. I said the assumption is it's massless so therefore it has to be taught. That's one of the reasons why we have to make a massless. If it wasn't massless, it could start getting kinks and I'll be doing this, but the massless one cannot, and that's a good question for you to think why that statement is correct. It's nontrivial. All right.

Then at some instant of time, this angle I'll called theta. And now this is a problem in dynamics. Basically, I can draw my free body diagram, a force diagram. I have this force, mass m. What forces are acting on it? Well, there is gravity pulling it down, the force mg. There is a frictional force which we assume will be proportional to the velocity, so this is the frictional force which will be minus b times the velocity.

There will be a tension in this string pulling it in that direction. And the sum of forces, by Newton's laws of motion, will give rise to the acceleration. The acceleration will be net force divided by C inertia, the system which is the mass. We have now experience. I'm not going to go slowly. I'll immediately take the components of forces.

First let me consider vertical motion. I told you the angle is small. If I make the angle sufficiently small and that ball is oscillating, it is not moving up and down. The distance is tiny insignificant in this problem. So I can ignore the vertical motion. So the vertical acceleration is 0. The net vertical force here is the component of t in the up direction which is t cosine theta minus the gravity down, and that's equal to 0, no acceleration. Therefore t is mg. T is the tension in this string. How about horizontal? For horizontal motion, I have the horizontal component of this tension here which is t sine theta. This is theta, so t sine theta is the horizontal component.

Now, this is moving here, therefore, there will be a frictional force proportional to that. So that's minus, this is the velocity, times b, so this is the frictional force. It's always minus, it's always in opposition to the actual velocity. And that by Newton's law is equal to mass times the horizontal acceleration.

Beware of my symbols. This whole problem for me is really in one dimension. So I called x the horizontal motion of my hand, and I'm calling y the horizontal motion of the mass.

This has nothing to do with x, y, z, one being up and the other horizontal. It's all horizontal. I had a choice. I could have called this x1 and this x2. I chose to call them x and y. You're smart enough to figure the difference, to follow that kind of terminology.

So now I can rewrite this. I know what sine theta is if I look at this diagram. Sine theta is simply equal to the distance. It's x minus y. It's sine theta. This angle equals, I should draw this here. This angle equals that one, so sine theta is this distance divided by that length, and this distance is x minus y over l.

So this is sine theta minus by equals this. Quickly, I can play with the algebra, and I end up with this equation. And lo and behold, Eureka, once again I get almost the same equation. I purposely chose it so it didn't come out exactly right. Before I ended up with a cosine, now I have ended up with a sine. Intention. All right.

So to show that something which seems different often is not. So rewriting this in terms of constant, which I like, I do the same as before. I define this b over m as gamma, this g over l as omega squared, this thing here as f, and I end up with this equation where the constants are here and the initial conditions are that the position of this mass at time 0 is 0, and that the velocity was 0. You remember I considered the mass being stationary.

This is now a set of mathematical equations. I have described this problem in terms of mathematics. From now on, I can forget that this has anything to do with that problem. I have to solve now a mathematical problem.

Finally. One more. Let me consider one more problem, and once we've done all three, at least part of step 1 of all three we'll go and solve them all in one go. And I repeat what I've done here I've done all the physics. This is the part which causes most difficulty, needs most understanding of the physical situation, et cetera. It may or may not take a very long time.

The next step is routine mathematics, which can be the part which takes you a whole night solving because you make algebraic errors, and you sweat and you go back, et cetera. That's the part you may be cursing when you're doing it, but it is routine from then on. It doesn't need an understanding of the physics of the situation, and what I'm trying to help you understand is physics, not mathematics at this stage.

So let's take another problem, which again, seems completely different. All right. Here it's a seismograph, in other words, a device for detecting tremors of the Earth. And the following would work. Take attached to the floor a spring. Put the mass, m, on it, and just look at the mass how it's behaving.

If the Earth starts oscillating, an earthquake, the mass will oscillate. How much it oscillates will give you a measure of how badly the Earth is oscillating. So I'm going to here do an idealized seismograph. I will model it by a mass, m, sitting on top of a massless, ideal spring, which is attached to the floor.

I will idealize the earthquake and saying it's oscillating with a single frequency with a single amplitude. The floor is oscillating by an amplitude y0 sine omega t. It's oscillating. So my assumptions are this spring is ideal massless.

The base, Hooke's law, that's the ideal part. I will again assume that there is a damping, which will be proportional to the velocity, for the same reasons as I took the other one to be proportional to velocity. And the question this time will be not to predict where that mass is at a given interval of time, but the question is, for what values of the mass and the spring constant, k, will I get the most sensitive instrument?

So this seems completely different and the question seems different the other ones et cetera, but as you will see it all boils down to one and the same question. So how do we do this? So now let me come down to step 1, understanding the situation, the problem in terms of mathematics.

All right. So I've re-drawn here the situation. This is my coordinate system of the problem. The surface of the floor, I'll assume it's here. It's moving. It is not a good reference place. It's moving. All right. So it's here. The mass, m, is there attached to the spring of natural length, l0, spring constant, k. There is the mass sitting on it.

Now, we've got to take a coordinate system which isn't moving. Got to have an inertial one. The surface of the Earth is no good for this. It's vibrating. So the best thing we have are the fixed stars. So let's take the fixed stars as a reference. So this is a height in that room, which is fixed in position relative to the fixed stars.

Now, the surface of the Earth can be moving relative to that, and I am making the assumption it's moving sinusoidally with amplitude y0. So this is the floor, position of the floor at time, t. The mass relative to this, which I defined as my y equal 0, is at position y of m. So this now is again a problem in dynamics, a mass with forces acting on it, what is the acceleration, and what is the solution of the dynamic equation, the equation of motion.

So the force diagram or free body diagram is a point mass, m, with a force fg due to gravity on it, a force, f, due to the spring acting on it, and I forgot to put this. Of course, there will be a force here due to friction acting on it. This is probably not a good place to draw it. Let me draw it so it doesn't confuse you. I'll draw it next to this, friction. So these are in same direction et cetera.

All right. So now let's use Newton's laws of motion, f equals ma, and relate the forces to the resulting acceleration. So first of all what is the net up-going force? I'm taking a coordinate system where this is a one dimensional problem, so I don't have to use vectors. I'll consider just the magnitude. My positive force is upwards.

So at any instant of time, this mass will have a force due to the spring and it will depend. If this spring is compressed, the force will be up. If it's elongated, stretched, the force will be down. And the magnitude would be k times the change of length from the natural length, so the upward force will be the difference between those two subtracted from the natural length.

So k times the natural length minus this quantity, which is the difference between this and this, which therefore this is how much the string is compressed, how much is compressed and is forced upwards. If this is a positive number, it's a negative one, it's stretched, it's putting it down. So this is the force up due to the spring, this is the force down due to gravity, and this is the force in opposition to the velocity. I don't know whether it's up or down. It depends which way it's moving due to the friction.

This is the net force, and that's got to be equal to the inertia of the system, which is the mass times the acceleration of it to the upward direction. And I told you this spring is massless, et cetera, so I don't have to take into account the motion of the spring. And that's why I have to take an idealized situation.

Back now to algebra. From here, I manipulate this, and I end up with this equation. You can do the same, and I've replaced for example, the motion of the floor by y0 sine omega t, et cetera, from here. And at first I say, oh, hell. This looks different than the one equation I've seen before.

But then I don't have to be an Einstein to realize, hold on. If I redefine this as my variable, in other words, if I take define y as ym plus mg over k minus l0, this quantity, the rate of change of y is the rate of change of ym because these are constants, and the second derivative of y is the second derivative of this, is a constant. And therefore, I can rewrite the equation as well y double dot plus gamma y dot plus omega 0 squared y is equal to a constant and sine omega t where these constants are written here from this equation and I redefine my variable.

And now, Eureka, I'm back again to the same problem. So once, again, what we've done. We've taken a physical situation and a physical question from our understanding of the laws of nature which apply to that situation. We have translated it to a problem in mathematics. And here it is. That's the problem, and I can now, at least for the next few minutes, forget that this is physics, treat it as a problem in mathematics.

And that's what I'll do it now, but first I've got to make some room for myself. I have nowhere to write, so I have to erase some things, and we'll continue in a few minutes.

All right. We've erased the boards, and we can therefore continue. Let me remind you what we have shown is we've taken three completely different systems, described them in terms of mathematics. The three systems with an electrical circuit, an RLC circuit. We took a pendulum, driven on a pendulum, and a seismograph. Each one of these ended up with almost identical equation. In fact, the only difference is one of them ended up with a cosine here, the other two with a sine. I will comment about that in a second.

We are now, forgetting about the physics, we are now in the world of mathematics. These equations of motion need to be solved if we are to predict what will happen in those situations. Now, there are many ways of solving differential equations. It is not my job to teach you how to solve differential equations.

For an example, you can go to the lectures of Professor Walter Lewin, and he does illustrate how one can solve these using complex amplitudes. I am satisfied if I find the solutions. I don't care how. I can take them from what he did, I can take them from a book of mathematics, et cetera, and what you will find is that if you have a generic second ordered differential equation of this form, it has solution which looks like this, the most general solution that looks like that.

There is no other solution in the universe which is not represented by this equation. You can manipulate it, make it look different, but it will boil down to this equation. How do I know that? Because of something I mentioned earlier, the so-called uniqueness theorem. This is a second order differential equation. This equation satisfies it.

Try it, take it, differentiate it twice this, et cetera, u [? dissatisfied. ?] Furthermore, it has two arbitrary constants. This C can be anything and this phi can be anything. They are arbitrary. Everything else in this equation is completely determined. For example, a of omega depends on omega. This amplitude, a, is equal to f divided by the square root of omega 0 squared minus omega squared 0 squared plus gamma omega 0 squared.

All these constants-- omega, gamma, f, omega 0-- are known in each of the problems. Let me just as an example take the last problem we did When we derived the equation of motion, we furthermore knew exactly what each one of those constants meant. They are known, so I don't need to solve for them. So a of omega is this quantity.

The delta, the phase there is, again, a known quantity. The time of delta is this. It is only under these conditions that this satisfies that equation. And find the omega prime here also is a known quantity. Nothing here is unknown. The only unknowns are the C, and the phi, and they are arbitrary constants. This works for any value of C and any value of phi.

Now, one more comment. These look equations. One looks like a cosine theta sine. If you stop and think for a second, they're actually one and the same equation where you've redefined what is t 0. So by just changing the phase of this, you'll get to this. You've changed it by 90 degrees. Just redefine what t0 is, your clock, and you'll get from this to that.

So if this equation I've written here is a solution to the cosine part, if my driver has this phase, in other words, sine omega t, then the answer will be displaced with respect to this one by that same thing. Instead of having a cosine here, you have a sign. So from my perspective, these two are identical equations, and the solution for them is identical.

It's also worth looking at what the mathematics of this looks like. What this looks like is the solution to these equations is some function with a certain amplitude, which is oscillatory with the same frequency as the driver. So this is just an oscillatory function. It's got a different phase to that. They lag will lead it, different phase, but it's just constant amplitude oscillatory function.

Added to it is another oscillatory function with a slightly different frequency, but which with time decays, because of this e to the minus gamma over 2t. And by the way, one more thing I should just emphasize. In writing this solution, I've made the assumption that this oscillator does oscil. You're driving something which oscillates and is an overdamped system in which case this term here would be different and not interesting when normally you wouldn't drive things and be interested in how it responds if it's incredibly damped system.

So I am making the assumption that this is an underdamped system, which means that this quantity here is greater than this quantity here. And if you're having difficulty with this part, I urge you to review what happens with just not driven oscillators under conditions of being underdamped or overdamped. That's the same physics.

So we found the most general solution. This solution must apply to every problem we've done. So I'll now take one of them. For example, let's consider this problem. We did this. We have an RLC circuit. They are an equation of motion.

From here, I can immediately write what is the solution of this. As a function of time, the charge Q will be equal to-- just look at that-- a of omega times cosine omega t minus delta-- I'm just copying from over there-- plus some arbitrary constant e to the minus gamma over 2t. And here we have a cosine, so I took the cosine here, and here is cosine of omega prime t plus this arbitrary phase phi.

So this is a general equation, which satisfies that one at all times. And I'm repeating myself. This, you know, it will be given by A of omega will be equal to f, and what is f? f is v0 over L-- I just read it off there-- divided by square root of omega 0 squared, which is 1 over LC minus omega squared

That's just the frequencies with which the voltage is driving there all squared plus gamma omega. Gamma is R over L omega all squared. And time delta is equal to gamma omega. Gamma is R over L divided by omega 0 squared, which is 1 over LC minus omega squared.

So far we know everything except the C and the phi. The question actually was, I've now erased, it but the question was what is the current flowing? And the current flowing I of t, is equal to dq dt with the way we define the signs with a part plus here, which is equal to Q dot, which I can get from this by differentiating this equation, and I get differentiate this minus omega A of omega sine of omega t minus delta.

Go to differentiate this, I get minus gamma over 2 Ce to the minus gamma over 2t times cosine omega prime t plus phi. So I've differentiated this times that. Now I've got to take this differentiated by that, which minus Ce e to the minus gamma over 2t times omega prime times sine omega prime t plus phi. Sorry. That's so.

So and again, we know every quantity here except for C and phi. Those are unknowns. So this would be the answer to my question if I knew C and phi, then everything is known. Then I could predict exactly what would happen with any instant of time. How do I find those two? This is where the boundary conditions come in.

That's why you need at least two boundary conditions or initial conditions, two facts about this system to be able to determine those constants, and we're always in the physical situation of this kind, you will have two bits of information. For example, in this case, I told you that we will assume that t equals 0 the charge and the capacitance is 0 and the current in the circuit is 0. I could have told you something else at the beginning and the answer would be different.

All right. And the thing that would be different would be those constants. So how do we do it? Simply by taking this equation and writing it at the time t equals 0. So at t equals 0, we know what Q of 0 is, and we know what Q dot of 0 is. So I take this equation, set t to 0, and write down this equal to 0.

Sorry. I take this equation, the Q. This one set t to 0 and write 0 equals that with t equals 0. I write another equation from here. At t equals 0, Q dot is 0, so I take this one, set t to 0, and write another equation. I will end up with two algebraic equations with two unknowns which you can solve.

Now remember I told you, the hard part conceptually is getting the equations of motion right. But the slog, the sweat, and a hard labor is getting this solution right, and I'm not going to go here now and embarrass myself making error after algebraic error solving those equations. I know you can solve them if you keep your cool and follow all the constants, et cetera.

You take these two algebraic equations, solve them, you come out with a value for C and a value for the phase phi, and then I have the final answer. I then have this equation I of t and Q of t for any T. And so I can predict at any time with this will do. End of story.

Now, let me take another case and to look at this one here, and I'll go a little faster. This is the mathematical description of the pendulum. So again, I'll take this, initially I'll do the same as I did before, so I will write that y at time t is equal to-- and I have to once again just follow the thing-- it's f, which is gx over L divided by the square root of omega squared, which is g over L minus omega squared all squared plus gamma, which is b over m this time gamma omega all squared.

So times. Now, here I have a sign so it's going to be the sine omega t minus delta plus-- and to save time, not rewrite it, the full thing, but I'll start at the beginning. There's the Ce to the minus gamma over 2t here at cosine et cetera.

So what we see here as before, we can get what this mass will do as a function of time, and as before, I don't want to just go around and around repeating the thing is. In order to solve, everything here is known except the C and this phase here-- maybe I should at least write that in. That is omega prime t plus phi. And in principle, this and that you can get, again, by using the initial conditions.

I don't want to focus on that. In this case, I want to focus what the problem said. What it said it was, discuss what you would expect the motion to be for that. In other words, when we took this special, we took this and took in given initial conditions and let's go. This will describe what goes on.

And let's look at this solution. What we see is that there are two terms here as before. This is an oscillation of the same frequency as a driver. So other words, that term is something which has this moving with the same frequency as my hand. This is a slightly different frequency, which is exponentially decaying.

When you add two oscillating functions, you've learned from Professor Walter Lewin's class, you get beats. So at first when this is large, these two wave motions, these oscillatory motions will beat minutes one against the other and you will get some kind of a crazy motion. You can't make sense of it.

As a function of time, as t becomes bigger and bigger, this exponential kills this term, and after a long time, you'll end up with only this term, which is simple sinusoidal motion. Try it for yourself. If I start any old way and I then move with a given frequency, you'll find this has a constant frequency, some amplitude given by this, and it continues doing this forever consistent with it.

But there is another interesting feature, which I would like you to focus on. If the friction is very small, like this, this will have very little damping, after a long time, this has gone and this term I can neglect. And what I see here is that this, the amplitude of the oscillations of this not only depends on the amplitude-- and again, I notice the time here; this is the node quantity here-- not only depends on the amplitude of how far my head is moving, but it depends critically on the difference between g over l and omega squared.

This, if you remember from the simple pendulum-- that's this omega 0 squared-- gives the frequency of oscillation of a free pendulum of this length. So what happens depends critically how different and of which magnitude is between the frequency of oscillation of my hand compared to the natural frequency of oscillation. If-- and here I won't do the algebra. I'll let you do this as an exercise-- play for yourself changing the value of this making it bigger than that or smaller than that.

And what you will find is that the motion of that and sorry, there's one more thing I should have said I didn't, and let me just say it now. There is also this question of this phase delta, tan delta is equal to gamma omega over omega 0 squared minus omega squared. So the sine of delta depends on whether it's bigger or smaller.

Play for yourself changing this value. See what happens to the amplitude here and what happens to this phase, and what you'll see is that if I go slowly, the phase is such that the two go in phase together if my oscillating frequency is lower than the natural frequency of oscillation of this, while if I go faster, in other words, higher frequency, this becomes a negative and the phase gets out of phase by 180 degrees, the two.

Furthermore, if you look at this amplitude, you can completely understand it in the following way. Imagine that I had this string and I move it slowly. I'm moving this slowly, and what you'll see is it's going in phase with that, and the angle of this will be such that this string appears to be the continuation of a single string, which has a longer length. And the string with the longer length has a lower frequency.

So if I move here with the lower frequency than the natural frequency, this will appear to be part of a string which has just the right length to give me the frequency of my hand. While if I move with the higher frequency, what the string will do is do this. Here is my hand moving backwards and forwards. There is the mass, and the mass is moving backwards and forwards. All right. This string is oscillating as if it was a free string of a shorter length.

I urge you as an exercise, to play with the numbers here and consider these two cases. It is a beautiful, educational tool to understand why if you have a driven system, response can be in phase or out of phase depending on the relative frequency you're driving the system compared to the natural frequency of oscillation of the oscillator. And this example brings everything out.

Take this example, consider very little damping so that you can do the calculations easily. See what's the relative amplitude of the driver compared to the response and what's the relative phase of the driver compared to the response, and all this characteristics you'll see which you can play for yourselves with a little model. I really urge you to do that.

Finally today, I will just comment. You see we're now in a world of repeating. We're doing the same thing over and over again. I would just like to comment about the seismograph. Once again, this is an equation of this form. It's the sine version.

So this is the solution to it with a sine, and the question is, in this case was, what the value of k over m makes it most sensitive? That's equivalent to asking the question, after a long time-- this earthquake has lasted a long time-- this so-called transient, this term dies away. So this we can forget.

This is the response of the mass to the earthquake which we assumed it's a uniform frequency, sinusoidal driver. And so the question is, what is the amplitude of the response? How big is this term? This term we know how big it is, and if you look at this function, you see that this amplitude depends on omega, on the driver.

Now let's assume the damping is not very big. This will be small compared to that. And so if you want a big response, you want this quantity to be very small so that f divided by this is a big number. When will this be small? When these two are almost equal. So what you want, if you want to have an instrument which is very sensitive, you want to study so the dominant frequency of oscillations of earthquakes in your area and adjust the natural frequency of oscillation close to that.

And the natural frequency is of course given by k over m. So what matters is, it doesn't matter what m is, what k is, but what's the ratio of k over m? You want that to be close to the natural frequency, that the natural frequency of this oscillator, that value k over m, to be close to the frequency of the oscillations of the earth in the earthquake.

If this driver really had a definite frequency, I could solve this problem to find the maximum. If I plotted this as a function of omega, I would get a curve something like this. This is a function of omega, the amplitude, the response of that oscillator. if you do that, you'll find that if the natural frequency of oscillation is over here, the peak is slightly at the lower frequency than the natural frequency of oscillation omega 0.

And so you could do this exactly by plotting it out to find out what is the optimum value of the frequency that you want to adjust so that you get the best response. So much for driven oscillators. We will now move to couple oscillators, and that we'll do next time, some problems. Thank you.